MATH 425 9/2/2022

Note Title 9/2/2022

Continuing our discussion of counting

How many different "words" can we make if we have n_1 letters of 1st kind

n_2 letters of 2nd kind

$\vdots$

n_r letters of rth kind?

$n = n_1 + \cdots + n_r$

Answer: $\dfrac{n!}{n_1! \cdot n_2! \cdot \dots \cdot n_r!} = \dfrac{(n_1 + \dots + n_r)!}{n_1! \cdot \dots \cdot n_r!} = \underbrace{\binom{n}{n_1 \; n_2 \; \dots \; n_r}}_{\text{multinomial coefficient}}$

(We are actually making words out of the letters $x_1, \dots, x_r$ when we are applying distributivity)

$(x_1 + x_2 + \dots + x_r)^n$

coefficient of $x_1^{n_1} x_2^{n_2} \dots x_r^{n_r}$

Example: Calculate

$$\binom{6}{\underbrace{3\ 2\ 1}}.$$

(should add up to 6)

Solution: $\dfrac{6!}{3!\,2!\,1!} = \dfrac{6\cdot 5\cdot \overset{2}{\cancel{4}}\cdot \cancel{3}\cdot \cancel{2}}{\cancel{3}\cdot\cancel{2}\cdot\cancel{2}} = \underline{\underline{60}}$

Another example: $\binom{8}{2\ 2\ 2\ 2} = \dfrac{8!}{2^4} =$ (

$$= \frac{\cancel{8}\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot \cancel{2}}{\cancel{2}\cdot\cancel{2}\cdot\cancel{2}\cdot\cancel{2}} = 7\cdot 360 =$$

$$= \underline{\underline{2520}}.$$

Important special case: $r=2$ binomial coefficient or combination number

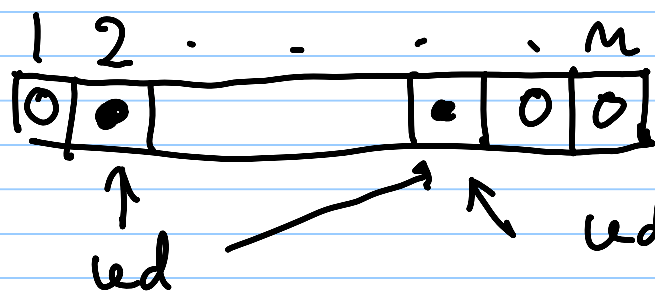
$$\binom{n}{k \quad n-k} = \frac{n!}{k!\,(n-k)!}$$

We wrote just. $\binom{n}{k} = \binom{n}{n-k}$

We discussed the interpretation as words, or, say, red and white balls.

Given k red balls, $n-k$ white balls, how many different ways can we arrange them in a row? (same color = interchangeable)

$$\binom{n}{k} = \binom{n}{n-k} = \frac{n!}{k!\,(n-k)!}$$



put red and white balls in these slots

Which slots contain red balls?

any k slots = any k-element subset of $\{1, \dots, n\}$

"n choose k"

$$\boxed{\text{The } \overset{\text{combinations}}{\text{number}} \binom{n}{k} \text{ counts } k\text{-element subsets of an } n\text{-element set}}$$

Remark: It also counts $(n-k)$-element subsets.

$$\binom{n}{k} = \binom{n}{n-k}.$$

Example: How many different 5-person committees can be chosen from people in a 40-person class?

Answer: $\binom{40}{5} = \frac{40!}{5! \cdot 35!} = \frac{\cancel{(1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \ldots \cdot 35)} \, 36 \cdot 37 \cdot 38 \cdot 39 \cdot 40}{(1 \cdot 2 \cdot 3 \cdot 4 \cdot 5) \cancel{(1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot 35)}}$

$$= \frac{36 \cdot 37 \cdot 38 \cdot \cancel{39}^{13} \cdot \cancel{40}}{1 \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdot \cancel{5}} = 36 \cdot 37 \cdot 38 \cdot 13$$

$$\binom{n}{k} = \frac{n\cdot(n-1)\cdot\cdots\cdot(n-k+1)}{k!}$$

k numbers on top

k numbers on bottom, counting 1.

$$\binom{n}{k} = \binom{n}{n-k}$$

$\leq n/2$

easier to calculate when $k \leq n/2$.

$$\binom{n}{n_1 \ n_2 \ \cdots \ n_r} = \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots$$

making words out of

n_1 letters 1

n_2 letters 2

$\vdots$

n_r letters r

where do I put the letters 1

where do I put the letters 2

$$\binom{n - n_1 - \dots - n_{r-2}}{n_{r-1}} \cdot$$

<u>Example</u>: Write

$$\binom{15}{4 \quad 3 \quad 3 \quad 3 \quad 2}$$

as a product of combination numbers.

Solution: $\binom{15}{4}\binom{11}{3}\binom{8}{3}\binom{5}{3} =$

$$= \binom{15}{2}\binom{13}{3}\binom{10}{3}\binom{7}{3} = \dots$$

(many answers correct).

The binomial theorem

$$\boxed{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}}$$

$0! = 1$

$\binom{n}{0} = 1 = \binom{n}{n}$

n times

$$\underbrace{(x+y)(x+y)\cdots(x+y)}_{n \text{ times}}$$

Distributivity: product = sum of "words" gotten by choosing x or y in every parenthesis

how many x's } k
and how many y's have } n-k
I chosen? $\binom{n}{k\;\; n-k} = \binom{n}{k}$ choices.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$y^0 = 1$ $\qquad$ $x^0 = 1$

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Pascal triangle

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$$\begin{array}{lccccccccc}
\binom{n}{0} & & & & & 1 & & & & \\
\binom{n}{1} & & & & 1 & & 1 & & & \\
\binom{n}{2} & & & 1 & & 2 & & 1 & & \\
\binom{n}{3} & & 1 & & 3 & & 3 & & 1 & \\
\binom{n}{4} & 1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

HW

5 Compute $\binom{9}{4\ 3\ 2}$

(answer should be a number, show how you got it)

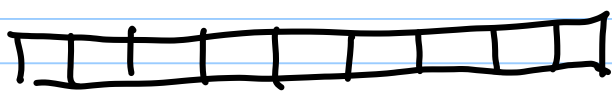
6 Write $\binom{18}{5\ 4\ 3\ 2\ 2\ 2}$

as a product of combination numbers.

7 Calculate $(x+y)^7$ using the Pascal triangle.

Why $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$?

n slots



did I put a red ball in the last slot or not?

I put k red balls in some of the slots.

If NO, I put the k red balls in the $n-1$ first slots

If YES, then I have
$k-1$ balls to put into
the first $n-1$ slots