MATH 425 9/7/2022

Note Title 9/7/2022

More examples with the binomial theorem (what if one of the variables is missing, or there are signs)?

$$1^k = 1$$

$$(-1)^k = \begin{cases} 1 & \text{if } k \text{ even} \\ -1 & \text{if } k \text{ odd} \end{cases}$$

**Example:** Calculate

**Note:** $\binom{90}{0} = \binom{90}{90} = 1$

$$1 - \binom{90}{1}4^1 + \binom{90}{2}4^2 - \binom{90}{3}4^3 + \cdots + 4^{90}$$

**Solution:** $(x+y)^{90}$ but what is $x$ and what is $y$?

$x = -1$, $y = 4$

$$(-1+4)^{90} = \underline{\underline{3^{90}}}$$

$\square$

Follow-up:

$$1 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

$$1 - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0.$$

$x = 1 \quad y = 1$

$x = -1 \quad y = 1$

$$\binom{3}{0} - \binom{3}{1} + \binom{3}{2} - \binom{3}{3} = 0$$

$$\underset{1}{\binom{4}{0}} - \underset{4}{\binom{4}{1}} + \underset{6}{\binom{4}{2}} - \underset{4}{\binom{4}{3}} + \underset{1}{\binom{4}{4}}$$

$$= 8 - 8 = 0.$$

Example: $1 + \binom{11}{1}\left(\frac{4}{3}\right) + \binom{11}{2}\left(\frac{4}{3}\right)^2 + \cdots + \left(\frac{4}{3}\right)^{11} = \underline{\underline{\left(\frac{7}{3}\right)^{11}}}$.

$x = 1 \qquad y = \frac{4}{3} \qquad (x+y)^{11} = \left(\frac{7}{3}\right)^{11}$

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Example: Flip a fair coin 8 times. What is the probability that H come up exactly 3 times?

**Solution:** $\binom{8}{3}\frac{1}{2^8}$ ← when do the H come up (3 out of 8 trials)

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**Follow-up:** Ask the same question with a rigged coin where the probability of H coming up is $\frac{2}{3}$.

**Solution:** $\binom{8}{3}\left(\frac{2}{3}\right)^3\cdot\left(\frac{1}{3}\right)^5$

$$P(T) = 1 - P(H) = 1 - \frac{2}{3} = \frac{1}{3}$$

Note (not needed for solution): replacing 3 by any number 0, ..., 8, it adds up to 1 by the binomial theorem:

$$\sum_{k=0}^{8} \binom{8}{k} \left(\frac{2}{3}\right)^{k} \left(\frac{1}{3}\right)^{8-k} = \left(\frac{2}{3} + \frac{1}{3}\right)^{8} = 1^{8} = 1.$$

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The <u>multinomial theorem</u>:

$$(x_1 + \cdots + x_k)^n = \sum_{\substack{n_1 + n_2 + \cdots + n_k = n \\ n_i \geq 0}} \binom{n}{n_1\ n_2\ \cdots\ n_k} x_1^{n_1} \cdot \ldots \cdot x_k^{n_k}$$

**Example:** $(x+y+z)^2$ using the multinomial theorem

$$
\begin{array}{llll}
2 = & 2+0+0 & \binom{2}{2\ 0\ 0} & = 1 \\
 & 0+2+0 & \binom{2}{0\ 2\ 0} & = 1 \\
 & 0+0+2 & \binom{2}{0\ 0\ 2} & = 1 \\
 & 1+1+0 & \binom{2}{1\ 1\ 0} & = 2 \\
 & 1+0+1 & & = 2 \\
 & 0+1+1 & & = 2
\end{array}
$$

$$(x+y+z)^2 = \underline{\underline{x^2+y^2+z^2+2xy+2xz+2yz}}$$

Follow-up question: How many summands do we have in the multinomial theorem?

In other words, how many ways can I write

$$n = \underbrace{n_1 + n_2 + \dots + n_k}_{\text{partitions}} \quad , \quad n_i \in \mathbb{N}_0 = \{0, 1, 2, \dots\} ?$$

Solution: Encode the partition $(n_1, \dots, n_k)$

by a sequence of 0s and 1s as follows "the lottery phone method"

$$\underbrace{1\,1\,1\,1\,1}_{n_1}\,0\,\underbrace{1\,1\,1\,1\,1\,1}_{n_2}\,0\,\cdots\cdots\,0\,\underbrace{1\,1\,1}_{n_k}$$

Examples of encoding

$6 = 4+1+1 \qquad (4,1,1)$

$1111010 1$

$3 = 2+0+1+0+0$

$(2,0,1,0,0)$

$$\begin{array}{ccccc} 2 & 0 & 1 & 0 & 0 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 11 & 0 & 0\,1 & 0 & 0 \end{array}$$

$1100100$

This way, we see every code corresponds uniquely to a partition and vice versa.

$$\boxed{\begin{array}{c}\#\text{ partitions} \\ \text{of } n = n_1 + \dots + n_k \\ 0 \text{ allowed!}\end{array}} = \#\text{ codes} = \binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$

1s, 0s

$n+k-1$

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**Example:** How many summands do we have if we express $(x+y+z)^7$ by the multinomial theorem?

**Solution:** $n = 7$, $k = 3$

$$\binom{n+k-1}{k-1} = \binom{9}{2} = \frac{9 \cdot 8}{1 \cdot 2} = \underline{\underline{36}},$$

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**Variants:**

(1) Partitions with no 0 allowed.

ways of expressing

$n = n_1 + \cdots + n_k \qquad n_i \in \mathbb{N} = \{1, 2, 3, \ldots\}$

Reducing to the previous problem:

$$n-k = (n_1 - 1) + \cdots + (n_k - 1)$$

$$n_i - 1 \in \mathbb{N}_0 = \{0, 1, \ldots\}$$

Answer: $\binom{n-k+k-1}{k-1} = \binom{n-1}{k-1}$

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Examples: There are 3 kinds of candy in a candy store. Alice is to choose 10 pieces of candy while she wants to try each kind at least once. How many possible choices does she have?

Solution: $n = 10$ $\qquad \binom{n-1}{k-1} = \binom{9}{2} = \underline{\underline{36}}$

$k = 3$

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Variant 2 (NOTA):

none of the above

There are $k$ candidates. $n$ people are allowed for them (but not required to).

How many possible vote distributions are there?

(partitions of a number $\le n$ into $k$ numbers)

To reduce it to the previous problem, introduce a dummy candidate called NOTA. whoever did not vote, voted, in fact, for NOTA. Replace $k$ by $k+1$.

answer $\underline{\underline{\binom{n+k}{k}}}$.

HW: (1) Casting a standard cubic fair die 8 times, what is the probability of 6 coming up exactly 5 times?

(2) $1 - \binom{12}{1} 3 + \binom{12}{2} 3^2 - \cdots + 3^{12} =$

(3) Express $(x+y+z)^3$ using the multinomial theorem

(4) How many monomials are there when expressing $(x+y+z+t)^{11}$ using the multinomial theorem?

(5) Pauline is to choose 8 pieces of candy in a store which has 5 kinds of candy. She must choose each kind at least once. How many different choices does she have?

(6) In a village, there are 100 inhabitants. In an election, there are 4 candidates for mayor. How many possible vote distributions are there?