MATH 425 9/19/2022

Note Title 9/19/2022

Introducing the concept of infinity to probability

Consider the example where we cast a fair coin and count the first time H come up.

Sample space:

never

$$\{1,2,3,\cdots\} \cup \{\infty\}$$

$$P(n) = \frac{1}{2^n} \qquad P(\infty) = 0.$$

This infinite set is called countable. This means you can index its elements by natural numbers

$\mathbb{N} = \{1, 2, 3, \cdots\}$.

$S = \{a_1, a_2, a_3, \cdots\}$

$a_1 = 0 \quad a_2 = 1 \quad a_3 = 2, \cdots$

Examples of Countable sets:

$\{2, 4, 6, 8, 10, \cdots\}$

$a_1 = 2, \ a_2 = 4, \ a_3 = 6 \cdots$

$\mathbb{Z} = \{ \cdots, -2, -1, 0, 1, 2, \cdots \}$

$0, 1, -1, 2, -2, \cdots$

$\mathbb{Q} = \{$ $-1, 0, 1$ rat. numbers ≤ 1 with denominator 1, $-2, -3/2, -1/2, 1/2, 3/2, 2$ rat. numbers x $|x| \leq 2$ with denom ≤ 2 not listed before, $\cdots \}$

$\uparrow$ rational numbers

all rational numbers can be listed in a sequence $\Rightarrow$ the set of rational numbers is countable.

For countable sample spaces, probability works the same as for finite ones. (There cannot be equal likelihood of outcomes.)

$$\sum_{s \in S} P(s) = 1$$

Infinite sums of countable sets of non-negative numbers can be always defined (using limits).

Examples:

$a > 0$

$$0 + 0 + 0 + 0 + \cdots = 0$$
$$a + a + a + a + \cdots = +\infty$$

equal likelihood not possible in countable infinite sample space

$0 < a < 1$

$$1 + a + a^2 + a^3 + \cdots = \frac{1}{1-a}$$

$$1 + a + a^2 + a^3 + \cdots =$$

$$= \lim_{n\to\infty} \underbrace{1 + a + a^2 + \cdots + a^n} = \frac{1 + \cancel{a} + \cdots + \cancel{a^n}\;\; - \cancel{a} - \cdots - \cancel{a^n} - a^{n+1}}{}$$

$$(1 + a + a^2 + \cdots + a^n)(1-a) = 1 + \cancel{a} + \cdots + \cancel{a^n} - \cancel{a} - \cdots - \cancel{a^n} - a^{n+1}$$

$$\lim_{n \to \infty} \frac{1 - a^{n+1}}{1-a} = \underline{\underline{\frac{1}{1-a}}}$$

$0 < a < 1$

$a = \frac{1}{2}$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right)$$

$$= \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{2}} = \frac{1}{2} \cdot \frac{1}{\frac{1}{2}} =$$

$$= \frac{1}{2} \cdot 2 = \underline{\underline{1}}$$

Example: An unfair coin comes up H with probability p $(0 < p < 1)$.
What is the probability when I keep flipping this coin that H comes up first in the nth try (resp. **never**)?

Solution: $S = \{1, 2, 3, 4, \ldots\} \cup \{\infty\}$

$$P(1) = \underline{p}$$

$$P(2) = \underline{\underline{(1-p)\,p}}$$

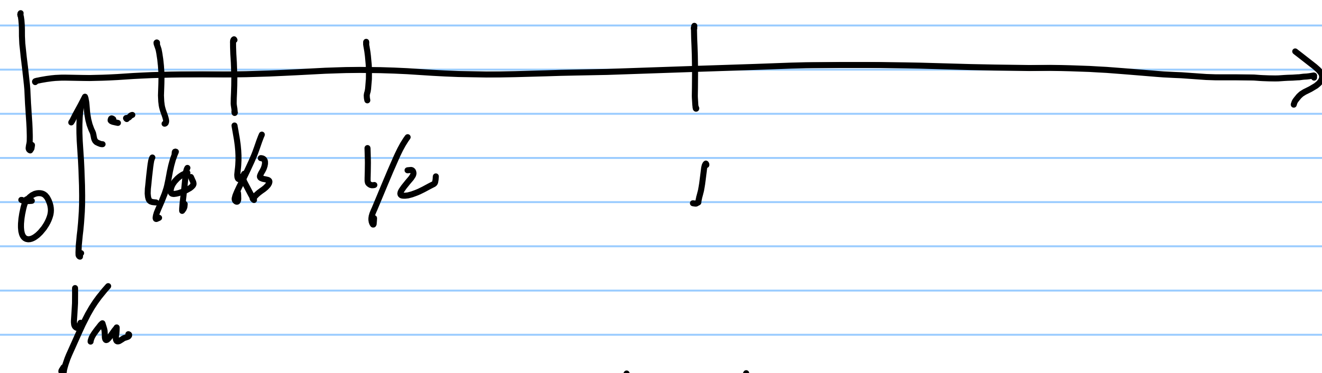
$$P(3) = \underline{\underline{(1-p)^2\,p}}$$

$$\alpha = 1-p$$

$$P(\infty) = 1 - \sum_{n=1,2,\dots} P(n) = 1 - \Big(p + (1-p)\cdot p +$$

$$(1-p)^2 p + \dots\Big) = 1 - p\left(\frac{1}{1-(1-p)}\right) = 1 - 1 = \underline{\underline{0}}.$$

Note that if S is <u>uncountable</u>, Σ does not work. A <u>sum</u> of uncountably many positive numbers is always ∞.



If S an <u>uncountable</u> set of positive numbers then for some n, infinitely many of them have to be $\geq 1/n$.

(if only finitely many of the numbers in S were $> 1/n$, then by increasing n, I would eventually get through all elements of S, so S would be countable).

$\therefore$ For an uncountable sample space,
we can no longer use Σ.

The real numbers (or any interval with >1 point)

are uncountable. $\Longrightarrow \int$

We can represent real numbers (between 0 and 1) by infinite decimal expansions:

$$0.a_1 a_2 a_3 a_4 \cdots \qquad a_n \in \{0, \dots, 9\}$$

(one caveat: $0.\overset{?}{\sim\sim}\,99999\cdots$ not allowed

$$0.1999999\cdots = 0.2\text{)}.$$

I cannot enumerate all real numbers

between 0 and 1 is a sequence. Suppose I could.

$$\left.\begin{array}{lllll} 0.a_{11} & a_{12} & a_{13} & a_{14} & \cdots \\ 0.a_{21} & a_{22} & a_{23} & a_{24} & \cdots \\ 0.a_{31} & a_{32} & a_{33} & a_{34} & \\ & \vdots & & & \\ & \vdots & & & \end{array}\right\}$$

suppose someone claims this sequence contains all real numbers between 0 and 1

$$0.b_1 \quad b_2 \quad b_3 \quad \cdots$$

They are lying! I can produce a new number

$0.b_1 b_2 \; b_3 \; b_4 \ldots$

where $b_n \neq a_{nn}$.

Example: $x_1 = 0.23145$

$x_2 = 0.14411$

$x_3 = 0.33219$

$x_4 = 0.61722$

$x_5 = 0.78134$

Produce a number $y = 0.b_1 b_2 b_3 b_4 b_5$ which differs from x_n in the nth digit.
$n = \{1, \ldots, 5\}$

$$y = 0.35317$$

(one of many correct solution).

HW ① Consider the numbers

$z_1 = 0.2217$

$z_2 = 0.3628$

$z_3 = 0.0419$

$z_4 = 0.1653$

Find a number t with 4 decimals $t = 0.b_1 b_2 b_3 b_4$ which differs from z_n in the nth decimal digit, $n = 1, 2, 3, 4$.

(Example of a wrong answer: 0.2561.)

(2) Suppose we keep casting a standard cubic die. Compute the probability that 6 will first come up in the n th try $n = 1, 2, 3, \ldots$ or <u>never</u>.