MATH 425 9/21/2022

Note Title 9/21/2022

Understanding such concepts as probability in relation to time.

<u>Example</u>: Suppose some event is "equally likely" to happen at any time $0 \le t \le 1$. ($t \in [0, 1]$)

closed interval

What is the probability it will happen exactly at time $t_0$?

Answer : $\underline{\underline{0}}$.  (If it were not 0, the same probability would arise for any $t$, the sum would be $+\infty$, but it is supposed to be 1.)

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By equally likely, we mean: if we pick a time interval $[a,b]$, the probability it will happen during that time interval only depends on its length.

If we are in the time interval $[0,1]$,

$P(E \text{ will happen in } [a,b]) = b-a$

other types of intervals

$(a,b]$
$[a,b)$
$(a,b)$

does not matter in this case

( ) mean bound not included

[ ] mean bound included

$$(a,b] = \{x \mid a < x \leq b\}$$

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But we do want to consider more general sets than intervals:

<u>Example</u>: If an event $E$ is "equally likely" to

happen at any time $t \in [0,1]$, what is the probability that it will happen at $0 < t < \frac{1}{2}$ or $\frac{2}{3} \leq t \leq \frac{3}{4}$?

Solution: $P\left(\left(0,\frac{1}{2}\right)\right) = \frac{1}{2}$

$$P\left(\left[\frac{2}{3},\frac{3}{4}\right]\right) = \frac{3}{4} - \frac{2}{3} = \frac{1}{12}$$

The intervals are disjoint so we can sum the probabilities:

$$P\left(\left(0,\frac{1}{2}\right) \cup \left[\frac{2}{3},\frac{3}{4}\right]\right) = \frac{1}{2} + \frac{1}{12} = \frac{7}{12}.$$

Example : There is a light which is red at time 0.
It flips to green at time $\frac{1}{2}$
Back to red at time $\frac{3}{4}$
back to green at time $\frac{7}{8}$ . . .
(The color changes at time $1-\frac{1}{2^n}$).

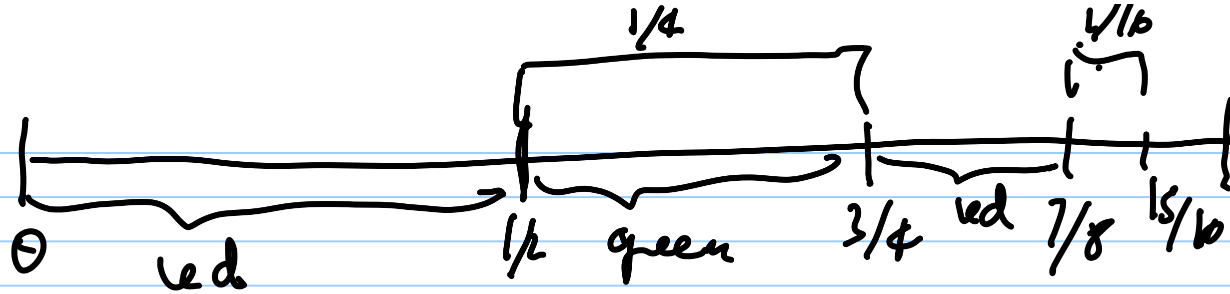
If I cross the street at a random time $[0,1)$

$[0,1]$

what is the probability that I will cross on green?

green green

**Solution:**




The length of the time intervals when the light is green are:

$$1 + a + a^2 + \cdots = \frac{1}{1-a}$$

$$\frac{1}{4}, \frac{1}{4^2}, \frac{1}{4^3}, \ldots \qquad \frac{1}{1-1/4} = \frac{4}{3}$$

**Answer:** $$\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots = \frac{1}{4}\overbrace{\left(1 + \frac{1}{4} + \frac{1}{4^2} + \cdots\right)}^{\frac{4}{3}}$$

$$a = 1/4$$

$$= \frac{1}{4} \cdot \frac{4}{3} = \underline{\underline{\frac{1}{3}}}\,.$$

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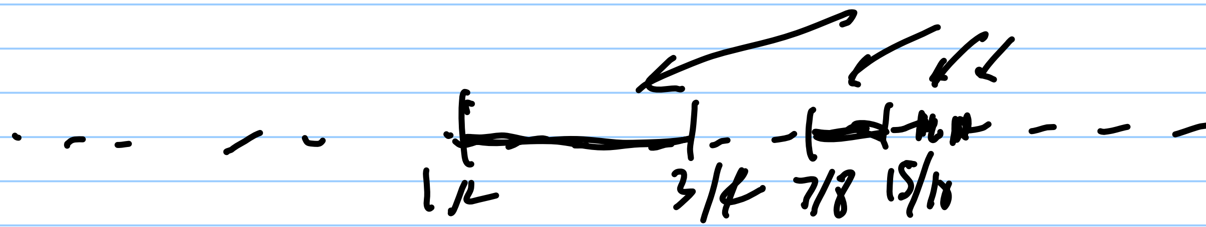
We have computed the "length" of a set which is a union of countably many intervals



Example: Suppose I choose a "random number" in $[0,1]$. What is the probability that the decimal expansion of the number does not contain the digit 2?

Solution: Each digit represents, according to its place, an interval of a given length: $\frac{1}{10}, \frac{1}{100}, \ldots \frac{1}{10^n} \ldots$

For a given place, the length of the interval does not depend on the digit.

Avoiding "2" in the place of 1/10s:

Probability $\frac{9}{10}$

Avoiding 2 in the place of 1/100s

$\frac{9}{10}$

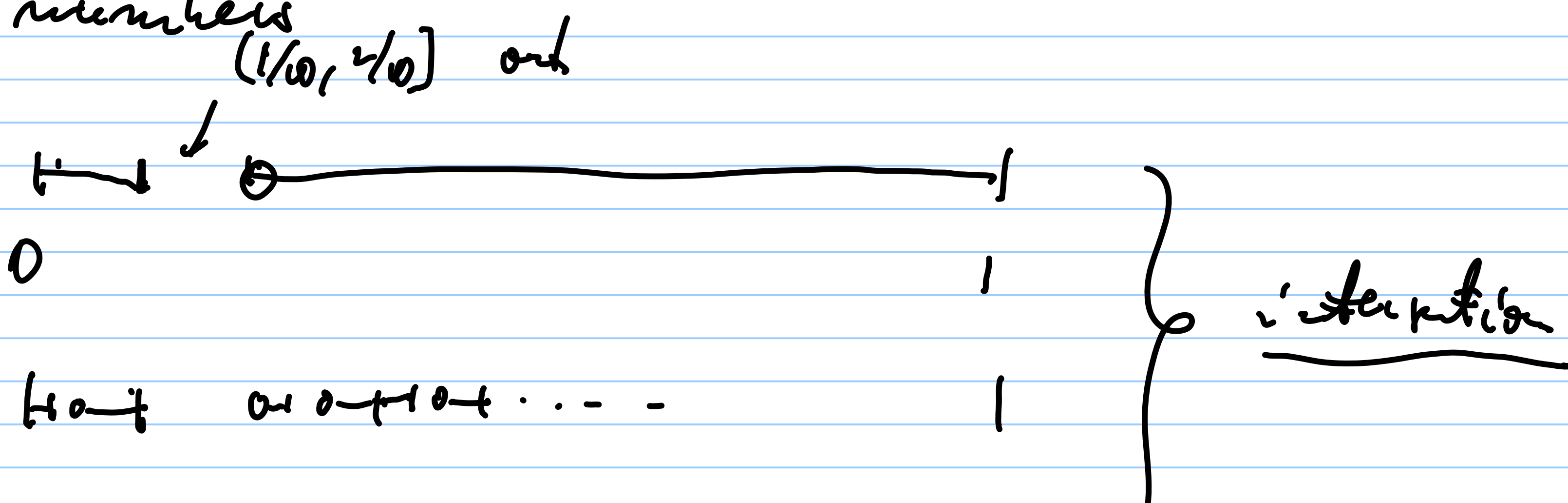
... Avoiding 2 in the place of 1/1000s

$\frac{9}{10}$

<u>Answer</u>: $\frac{9}{10} \cdot \frac{9}{10} \cdot \ldots \cdot \frac{9}{10} \cdot \ldots =$

$$= \lim_{n \to \infty} \left(\frac{9}{10}\right)^n = \underline{\underline{0}}$$

We calculated the "length" of another set of real numbers




(a)

This is called the Cantor set: we calculated its "length".

contains no interval

that's not only the
length was 0.
The length was given
by an infinite product,
and there are infinite
products of numbers between
0 and 1 which are not 0.

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1 < \infty$$

$$2^{-1/2} \cdot 2^{-1/4} \cdot 2^{-1/8} \cdot \cdots\cdots = 2^{-\left(\frac{1}{2}+\frac{1}{4}+\cdots\right)} = 2^{-1} = \frac{1}{2}$$

This $\underbrace{\text{fancier}}_{}$ notion of length is called measure.

applies to more different kinds of sets.

The notion of measure applies to $\underset{\text{MEASURABLE}}{}$measurable sets.

If $S$ is our sample space, then $S$ is measurable

(1) If $E_1, E_2, E_3, \ldots$ are measurable sets in $S$, (events)

then so is $E_1 \cup E_2 \cup \ldots = \bigcup_{n=1}^{\infty} E_n \leftarrow$ the outcomes what are at least in one

$S \setminus E$

of the sets $E_n$

(2) The complement of a measurable set $E$ is measurable.
(event)

$S \setminus E = \{$ all the outcomes not in $E \}$

complement.

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A **measure** assigns to each measurable set $E \subseteq S$ a number $\mu(E) \in [0, \infty]$.

If $E_1, E_2, \ldots$ are disjoint sets in $S$

↑ they share no outcomes

then $\Pr(E_1 \cup E_2 \cup \cdots) = \sum_{n=1}^{\infty} \Pr(E_n)$.

A probability is such a measure $P$ that $P(S) = 1$.

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Example: What is the probability that a randomly chosen real number in the interval $[0, 5]$

will be in the interval $[\frac{3}{2}, 2]$?

<u>Note</u>: The measure given by length is called <u>Lebesgue measure</u> (denoted by $\mu$).

$\mu([0,5]) = 5.$ This measure is <u>not</u> a probability.

But we do get a probability on the sample space $[0,5]$

$$P(E) = \frac{\mu(E)}{5}.$$

<u>Answer</u> : $\dfrac{m([3/2,2])}{5} = \dfrac{1/2}{5} = \underline{\underline{\dfrac{1}{10}}}.$

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HW (3) Suppose a light at an intersection is red at time 0 and changes to green at time $\frac{2}{3}$. It will change in time $1-\frac{1}{3^n}$. If I cross at a random time in $[0,1]$, what is the probability I cross on green?

(probability = Lebesgue measure)

(4) What is the probability that a randomly chosen number in the interval [0,10] is in the interval [2,3] or the interval [1/3, 1/2]?

(probability = a multiple of the Lebesgue measure.)

(5) Evaluate the infinite product:

$$2^{-1/5} \cdot 2^{-1/5^2} \cdot \ldots \cdot 2^{-1/5^n} \cdot \ldots$$