MATH 425 10/19/2022

If X is a $\underbrace{\text{discrete random variable}}_{\text{countably many values}}$ the expectation $E(X) = \sum_x x \cdot P(X = x)$.

idea: reduce the sample space the set of values

Example: If I keep flipping a fair coin, what is the average number of tries needed for H to come up?

<u>Solution</u>: $X = n$ where H first comes up in the n'th trial

$$P(X = n) = \frac{1}{2^n}$$

$$E(X) = \sum_{n=1}^{\infty} n \cdot \frac{1}{2^n} = \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} + 4 \cdot \frac{1}{16} + \cdots$$

$$\begin{aligned} &= \left.\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \right\} \frac{1}{2} \cdot 2 \\ &\quad \left. + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \right\} \frac{1}{4} \cdot 2 \\ &\quad \left. + \frac{1}{8} + \frac{1}{16} + \cdots \right\} \frac{1}{8} \cdot 2 \\ &\quad + \cdots = 1 + \frac{1}{2} + \frac{1}{4} + \cdots = \boxed{2} \end{aligned}$$

$$\boxed{1 + \frac{1}{2} + \frac{1}{4} + \cdots = \frac{1}{1 - \frac{1}{2}} = 2}$$

Some situations lead to problems:

Two "friends" play a game. They are flipping a fair coin, and betting money on H to come up. If H come up on the first trial, Friend 1 wins. If H first come up on the second trial, Friend 2 wins (and so on - they take turns). On the first trial, the bet is \$1, and on each subsequent trial it doubles. What is

(on trial 2 : TH $\Rightarrow$ F2 wins \$2
on trial 3 : TTH $\Rightarrow$ F1 wins \$4 ...)

the expected amount F1 wins? (or loses)

losses = negative winnings

X = amount won by F1 (could be < 0)

$$E(X) = 1 \cdot \frac{1}{2} - 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} - 8 \cdot \frac{1}{16} + \cdots$$

$$= \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \cdots \quad = \underline{\underline{??}}$$

The expectation <u>does not exist</u>.

A related concept is a <u>martingale</u>. You bet so much as to recover previous losses (and still win). With unlimited credit, you would always win.

A variant of the 2 friend game: The bets don't double.
The rules are the same as before, but whoever wins wins just $1. What are the expected winnings of F1?

Solution: X = Winnings of F1

$$E(X) = 1 \cdot \frac{1}{2} - 1 \cdot \frac{1}{4} + 1 \cdot \frac{1}{8} - 1 \cdot \frac{1}{16} + \cdots =$$

$$= \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots = \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{4} - \frac{8}{8} + \cdots\right) =$$

$$= \underline{\underline{\frac{1}{3}}}.$$

$$\sum_{n \geq 0} \left(-\frac{1}{2}\right)^n = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$$

Notice that if f is an ordinary real function and X is a discrete random variable, then $f(X)$ is a discrete random variable.

$$Ef(X) = \sum_x f(x)\, P(X=x),$$

Example: X is a discrete random variable.

$$P(X=-\tfrac{1}{2}) = \tfrac{1}{3} \qquad P(X=\tfrac{1}{2}) = \tfrac{1}{3} \qquad P(X=2) = \tfrac{1}{3}.$$

Calculate $E(e^X)$.

Solution: $E(e^X) = \frac{1}{3} \cdot e^{-\frac{1}{2}} + \frac{1}{3} e^{\frac{1}{2}} + \frac{1}{3} e^{2}$.

Variance and standard deviation

$$\mathrm{var}(X) = E\left(\left(X - E(X)\right)^2\right)$$

(sum of squares in math has the geometrical meaning of square distance).

Why square? Fair enough:

$$\sigma(X) = \sqrt{\mathrm{var}(X)}$$

$\nwarrow$ standard deviation (same "units" as X)

$$E(X - E(X)) =$$
$$= E(X) - EE(X) =$$
$$= E(X) - E(X) = 0.$$

$$\mathrm{var}(X) = E\left((X - E(X))^2\right) =$$

$$= E\left(X^2 - 2X\,E(X) + E(X)^2\right)$$

$$= E(X^2) - 2\left(E(X)\right)^2 + \left(E(X)\right)^2 = \underline{\underline{E(X^2) - \left(E(X)\right)^2}}$$

(note: we proved that

$$E(X^2) \geqslant \left(E(X)\right)^2.$$

<u>Cauchy inequality.</u>)

Example: Let X be a random variable with

$$P(X=-1)=0.2 \qquad P(X=1)=0.3 \qquad P(X=0)=0.5$$

Calculate $E(X)$, $var(X)$, $\sigma(X)$,

Solution: $E(X) = 0.2\cdot(-1) + 0.3\cdot 1 + 0.5\cdot 0 = \underline{\underline{0.1}}$

$E(X)^2 = 0.01$

$E(X^2) = 0.2\cdot 1 + 0.3\cdot 1 + 0.5\cdot 0 = 0.5$

$var(X) = \underline{\underline{0.49}}$ $\qquad \sigma(X) = \sqrt{var(X)} = \underline{\underline{0.7}}$,

$= 0.5 - 0.01$

HW: 1. Suppose the "friend" game is modified so that the person winning in the nth trial wins $\$\left(\frac{3}{2}\right)^n$.
Let X be the winnings of Friend 1. Does $E(X)$ exist? If so, calculate it.

2. Casting a fair cubical die, what is the average number of trials needed for 6 to come up?

3. Let X be a random variable with $P(X=0)=\frac{1}{4}$, $P(X=-2)=\frac{1}{4}$, $P(X=1)=\frac{1}{2}$. a) Calculate $E(X)$, $var(X)$, $\sigma(X)$

(b) Calculate $E(\cos(X))$

in radians