MATH 425 10/21/2022

<u>Remark</u>: Jointly distributed random variables $X_1, \dots X_n$ are random variables on the same sample space S. We say that $X_1, \dots, X_n$ are <u>independent</u> if for any choice of values $x_1, \dots, x_n$,

$$X_1 \le x_1 \ , \ X_2 \le x_2 \ , \ \dots \ , \ X_n \le x_n$$

are independent events.

We will now discuss a few distributions of discrete random variables which occur in many examples.

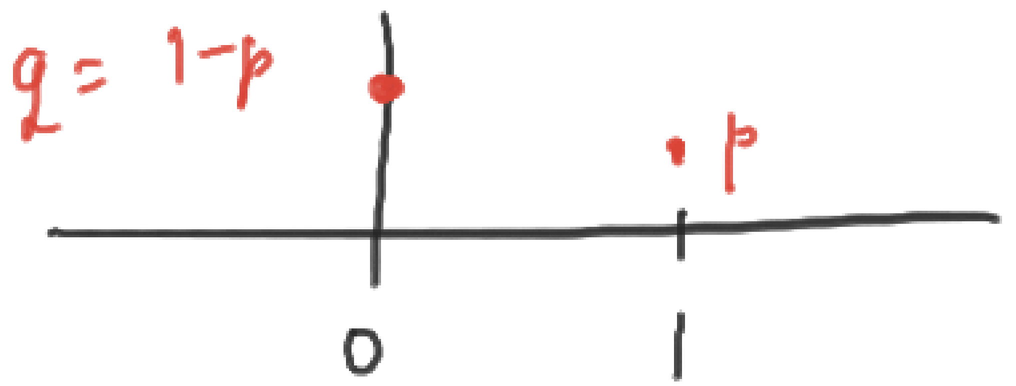
A Bernoulli variable X_p takes on values 0, 1.

(not a standard notation)

$$P(X_p = 1) = p$$

$$P(X_p = 0) = 1 - p \quad (= q)$$

Probability mass function graph:



$$E(X_p) = p$$

$$\operatorname{var}(X_p) = p(1-p) = pq \qquad \sigma(X_p) = \sqrt{p(1-p)} = \sqrt{pq}$$

The work for variance:

$$E(X_p^2) = p, \quad \operatorname{var}(X_p) = E(X_p^2) - E(X_p)^2 = p - p^2.$$

A binomial variable $X_{n,p}$ is a variable distributed the same way as a sum of n independent Bernoulli variables with the distribution of X_p.

(n Bernoulli trials ← probability p, add up the results "with replacement")

values of $X_{n,p}$: $\{0, 1, \dots, n\}$

$$P(X_{n,p} = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \binom{n}{k} p^k q^{n-k}$$

probability mass function

expectation is additive

$$E(X_{n,p}) = np$$

variance additive for independent variables

$$\mathrm{var}(X_{n,p}) = np(1-p) = npq$$

"probabilistic Pythagorean theorem"

$$\sigma(X_{n,p}) = \sqrt{np(1-p)} = \sqrt{npq}$$

Cumulative distribution of the binomial variable $X_{n,p}$:

$$P(X_{n,p} \le k) = \underbrace{\sum_{i=0}^{k} \binom{n}{i} p^i q^{n-i}}_{\text{There is no better exact formula!}}$$

(what we do)

Preview: If $n >> 0$, if we know E, σ, we can approximate by the normal distribution.

<u>Example</u>: Suppose we cast a standard cubic die 10 times.

(a) What is the probability 6 comes up exactly k times?

(b) What is the average number of times 6 comes up and what is the standard deviation?

ⓒ What is the probability 6 comes up 8 or fewer times?

Solution: $X_{n,p}$ $\quad n = 10$, $p = \frac{1}{6}$. $\quad q = \frac{5}{6}$

ⓐ $P(X_{10,\frac{1}{6}} = k) = \binom{10}{k}\left(\frac{1}{6}\right)^k\left(\frac{5}{6}\right)^{10-k}$

ⓑ $E(X_{n,p}) = np = \frac{10}{6}$

$\sigma(X_{n,p}) = \sqrt{npq} = \sqrt{10\cdot\frac{1}{6}\cdot\frac{5}{6}} = \frac{\sqrt{50}}{6} = \frac{5}{6}\cdot\sqrt{2}$

ⓒ $P(X_{10,\frac{1}{6}} \leq 8) = \sum_{i=0}^{8}\binom{10}{i}\left(\frac{1}{6}\right)^i\left(\frac{5}{6}\right)^{10-i} = 1 - P(X_{10,\frac{1}{6}} > 8)$

$\binom{10}{9} = \binom{10}{1} = 10$

$= 1 - \binom{10}{9}\left(\frac{1}{6}\right)^9\left(\frac{5}{6}\right) - \binom{10}{10}\left(\frac{1}{6}\right)^{10} = \boxed{1 - 10\frac{5}{6^{10}} - \frac{1}{6^{10}}}$

What about without replacement?

Hypergeometric variable

Scenario: N balls m are painted red $N-m$ are painted green.

Choose a sample of n balls (without replacement)

$$X^N_{m,n} = \text{The number of red balls chosen.}$$

values: $\{0, \ldots, \min(m,n)\}$.

$$P(X^N_{m,n} = i) = \frac{\binom{m}{i}\binom{N-m}{n-i}}{\binom{N}{n}}$$

$\boxed{\text{$n$ and m play symmetrical roles}}$

$$= P(X^N_{n,m} = i) = \frac{\binom{n}{i}\binom{N-n}{m-i}}{\binom{N}{m}}$$

Statistics : $E(X^N_{m,n}) = \frac{mn}{N}$

(actually it does not matter whether you are with replacement or not, when it comes to E)

$$\mathrm{var}\left(X^N_{m,n}\right) = np(1-p)\left(1-\frac{n-1}{N-1}\right)$$

$p = \frac{m}{N}$

not on my exam!

If n stays the same, p stays the same
$N \to \infty$, the hypergeometric distribution approaches the binomial

HW 4 If in a game I have probability 0.6 of scoring a point in one round and there are 15 rounds.

a) Name the distribution of the total number of points scored (with all parameters)

b) What is the average number of points scored and what is the standard deviation?

c) What is the probability I score ≤ 12 points?

5. Choose 4 cards out of a standard deck of 52.

X counts how many of the chosen cards are spades.

a. Name the distribution of X (with all parameters)

b. What is the average number of cards that will be spades?