MATH 425 10/24/2022

Poisson variable Y_λ

Poisson distribution – a certain limit of the binomial distribution

$n \to \infty$

$\lambda = np$ stays constant

$E(X_{n,p}) = np$

Values of Y_λ : $\{0, 1, 2, \ldots\} = \mathbb{N}_0$

$$P(Y_\lambda = k) = \lim_{\substack{n\to\infty \\ \lambda = np}} \binom{n}{k} p^k (1-p)^{n-k} = \lim_{n\to\infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

$$= \lim_{n\to\infty} \frac{n(n-1)\cdots(n-k+1)}{k!} \frac{\lambda^k}{n^k} \left(1-\frac{\lambda}{n}\right)^{n-k} = \lim_{n\to\infty} \frac{\lambda^k}{k!} \left(1-\frac{\lambda}{n}\right)^n = \frac{\lambda^k}{k!} e^{-\lambda}$$

$e^{-\lambda}$

$$\boxed{P(Y_\lambda = k) = \frac{\lambda^k}{k!} e^{-\lambda}}$$

Note

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{\lambda}$$

$$\lim_{x\to\infty} x \ln\left(1-\frac{\lambda}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1-\frac{\lambda}{x}\right)}{\frac{1}{x}}$$

L'Hôpital (take derivatives)

$$= \lim_{x\to\infty} \frac{\frac{\lambda}{\cancel{x}}\left(\frac{1}{1-\frac{\lambda}{x}}\right)}{-1/\cancel{x}^2} = -\lambda$$

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{\lim_{x\to\infty} x \ln\left(1-\frac{\lambda}{x}\right)} = \underline{\underline{e^{-\lambda}}}$$

Statistics:

$$E(Y_\lambda) = \lambda$$

$$\operatorname{var}(Y_\lambda) = \lambda$$

$$\sigma(Y_\lambda) = \sqrt{\lambda}$$

$$\lim_{n\to\infty} np\underline{\underline{(1-p)}} = \lambda$$

$$np = \lambda$$

Think of this as a binomial distribution with $n \gg 0$
$np = \lambda$.

- Number of calls to a customer service center in a given period of time.

earthquakes, accidents, any type of casualty assuming constant rate.

- Number of crimes which occur in a neighborhood in a fixed period of time.

Example: Suppose the average number of typos on a page of a book is 0.5. If I open a page randomly, what is the probability there will no typo.

Solution: $Y_{0.5}$ $\quad P(Y_{0.5} = k) = e^{-0.5} \cdot \frac{(0.5)^k}{k!}$ $\quad\Big|\quad 0! = \underline{1}$

$$P(Y_{0.5} = 0) = \underline{\underline{e^{-0.5}}} \approx \underline{\underline{0.39}}$$

Follow-up: Probability there will be at least one typo:

$$1 - P(Y_{0.5} = 0) = 1 - e^{-0.5} \approx \underline{\underline{0.61}}$$

Follow-up: A Chapter in the above referenced book has 10 pp.
What is the probability there is no typo in this chapter?

Solution 1: typos on 10 pp. independent: Answer $(e^{-0.5})^{10} = \underline{\underline{e^{-5}}}$

(There is a typo in the chapter with probability $\underline{\underline{1-e^{-5}}}$.)
at least one

Solution 2: The number of typos on 10 pp. is also a Poisson variable with a scaled expectation: $10 \cdot \lambda = 10 \cdot 0.5 = \underline{\underline{5}}$

Note: This method also works for a fractional number.

$$P(Y_5 = 0) = \underline{\underline{e^{-5}}}.$$

<u>Follow-up</u>: A chapter in the book has 8.5 pp. What is the probability there is at least one typo?

<u>Expectation</u>: $8.5 \cdot \lambda = 8.5 \cdot 0.5 = 4.25$

<u>Answer</u>: $\underline{\underline{1 - e^{-4.25}}}$

<u>Follow-up</u>: What is the probability there are at least 2 typos in the 8.5 pp chapter?

<u>Expectation</u>: $\lambda = 4.25$ $\quad P(Y_{4.25} = k) = e^{-4.25} \cdot \frac{(4.25)^k}{k!}$

<u>Answer</u>: $1 - P(Y_{4.25} = 0) - P(Y_{4.25} = 1) = 1 - e^{-4.25} - e^{-4.25} \cdot 4.25 = \underline{\underline{1 - e^{-4.25} \, 5.25}}$

An inverse problem: In another book, the probability of there being at least 1 typo on a page is 0.1. What is the average number of typos on a page?

Solution: $P(Y_\lambda \geq 1) = 0.1$? λ

$$P(Y_\lambda = k) = e^{-\lambda}\frac{\lambda^k}{k!}$$

$$P(Y_\lambda = 0) = 0.9 = e^{-\lambda}$$

Note: $\ln 1 = 0$

$\ln 0.9 < 0$

$$e^{-\lambda} = 0.9$$
$$-\lambda = \ln 0.9$$
$$\lambda = -\ln 0.9$$

Answer: $-\ln 0.9$

Example: An insurance company knows that the probability of there being a car wreck with total damage in 1 day is 0.1. What is the average number of total wrecks in a year?

Solution: $P(Y_\lambda \geq 1) = \overset{p}{\widetilde{0.1}}$ ← same numbers as in last problem

$\lambda = -\ln 0.9$

$-\ln(1-p)$

Assuming a year has 365 days,

Answer: $365\lambda = \underline{\underline{-365 \ln(0.9)}}$.

HW 1 A book has probability 0.99 of <u>not</u> having a typo on a page. What is the average number of typos on 10 pp. of the book?

2 An island gets an average of 0.4 earthquakes per year. What is the probability there <u>will</u> be an earthquake on a given day?