MATH 425 10/28/2022

Topics on Test 2:

conditional probability
- conditional probability, "two kids" examples and similar
- Bayes formula, also with multiple cases, reverse conditioning
- independent events (2 or more), "1st year / 2nd year insurance problem"
- odds

discrete random variables
- probability mass function and cumulative distribution
- statistics : E, var, σ
- function applied to a random variable
- Bernoulli
- Hypergeometric
- Binomial
- Poisson
- Negative binomial (includes geometric)

α any number, $k = \{0, 1, 2, \dots\}$

$$\binom{\alpha}{0} = 1$$

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdot(\alpha-2)\cdots(\alpha-k+1)}{k!}$$

$N_{r,p}$: # of independent Bernoulli trials of probability p needed for r successes. We write

$K_{r,p} = N_{r,p} - r$

We will use it for $\alpha = -1, -2, \dots$

$$\binom{n-1}{r-1} = \binom{n-1}{n-r} = \frac{(n-1)(n-2)\cdot\ldots\cdot r}{(n-r)!} =$$

$$= (-1)^{n-r}\frac{(-r)(-r-1)\cdots(1-n)}{(n-r)!} = (-1)^{n-r}\binom{-r}{n-r} = (-1)^k\binom{-r}{k}$$

k value of $K_{r,p}$ $\quad k = n-r$

$$\boxed{P(K_{r,p} = k) = \binom{-r}{k} p^r (-q)^k \qquad q = 1-p}$$

$\Leftarrow$ not on the test

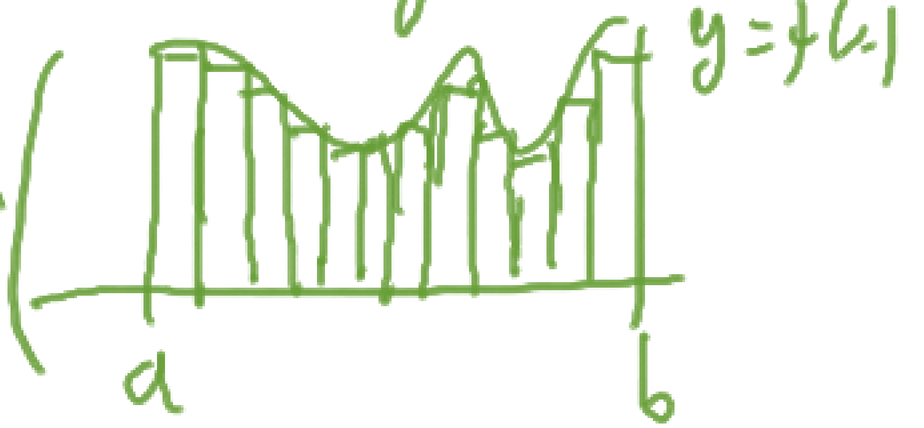
Continuous random variables

We say that a random variable X is <u>continuous</u> if there exists a function $f(x)$ such that

$$\underbrace{P(X \le x)}_{\text{cumulative distribution } F(x)} = \int_{-\infty}^{x} f(x)\,dx.$$

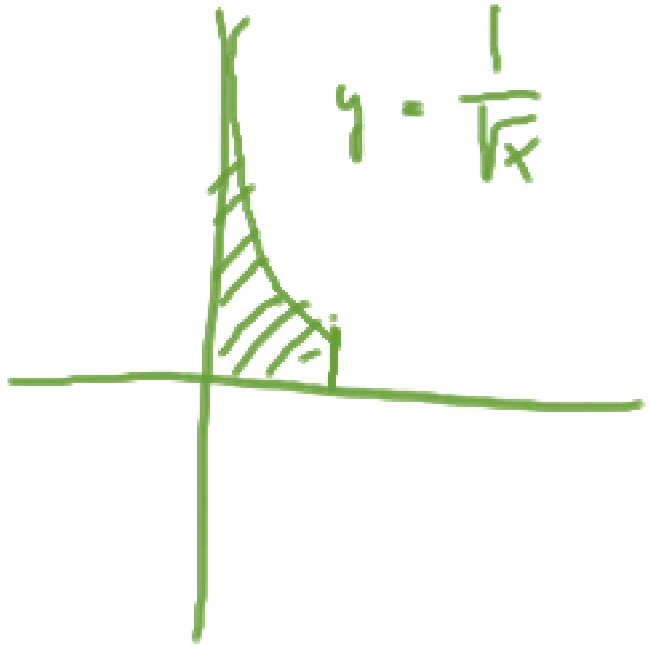
The function $f(x)$ is called the <u>density</u>.

Side note: The Riemann integral is not general enough for probability.

$\displaystyle\int_a^b f(x)\,dx = \lim \Bigg($  $\Bigg)$

The Riemann integral definition does not work with infinite bounds (or functions approaching ∞).

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_0^1 = \underline{\underline{2}}$$



Patching: "improper" integrals. How do they transform?
Do they satisfy substitution?

Lebesgue (the guy who invented Lebesgue measure) had the great idea of slicing the function horizontally instead of vertically.

$$\int_a^b f(x)\,dx = \lim \Big($$  $$\Big)$$

the horizontal slices are $\underbrace{\text{measurable set}}_{\text{apply measure}} \times$ interval

This solved all the theoretical problems with the integral.
It is called the Lebesgue integral.

Back to continuous random variables

If X is a continuous random variable,

$$P(X = x) = 0 = \int_x^x f(t)\,dt$$

$$\left(= \lim_{y \to x^+} \left(\underbrace{P(X \leq y) - P(X \leq x)}_{\int_x^y f(t)\,dt} \right) \right)$$

There is no probability mass function for continuous random variables, the density replaces it.

A continuous random variable gives rise to a sample space reduction to $\mathbb{R}$. This is the reason why we study geometric problems in probability.

A brief review of calculus:

$$(x^n)' = n x^{n-1}$$

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

except when $n = -1$

$$\int \frac{1}{x} \, dx = \ln|x| + C$$

Chain rule:

$$(f(g(x)))' = f'(g(x)) \, g'(x)$$

Integral substitution: $\quad d\,g(x) = g'(x)\,dx$

$$\int_{g(a)}^{g(b)} f(g(x))\,d g(x) = \int_a^b f(t)\,dt$$

$t = g(x)$

true when g is increasing (or decreasing)

$$(e^x)' = e^x$$

$$(\sin x)' = \cos x$$

$$(\cos x)' = -\sin x$$

$$(\operatorname{arctan} x)' = \frac{1}{1+x^2}$$

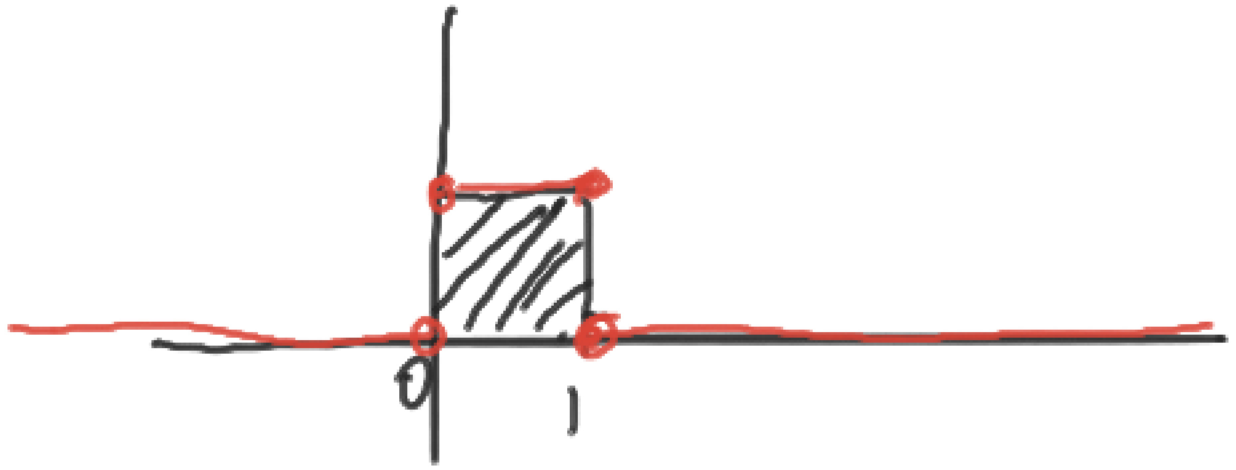
Statistics of continuous random variables:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \quad \left(= F(\infty)\right)$$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

<u>Example</u>: Let X be a continuous random variable with density

$$f(x) = \begin{array}{ll} 1 & 0 \le x \le 1 \\ 0 & \text{else} \end{array}$$



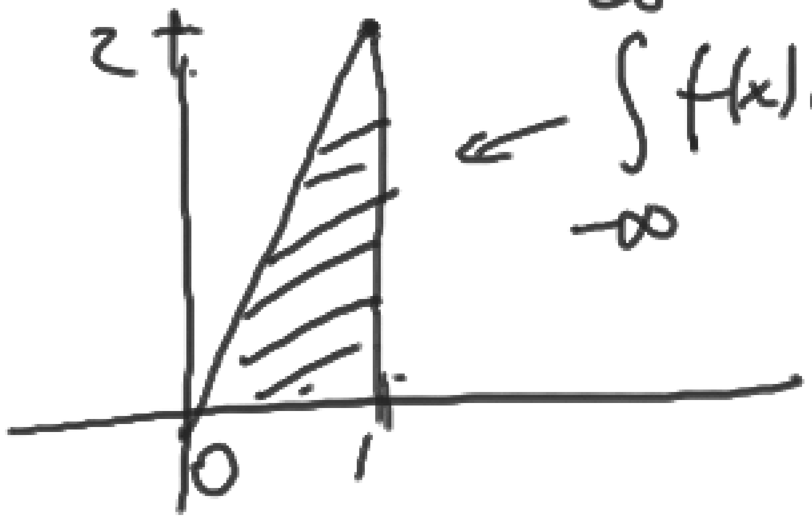
Calculate $E(X)$.

Solution :

$$\int_0^1 x \cdot f(x)\, dx = \int_0^1 x \, dx = \left[\frac{x^2}{2}\right]_0^1 = \underline{\underline{\frac{1}{2}}}$$

HW 4 Let X be a continuous random variable with density

$$f(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$$



$$\int_{-\infty}^{\infty} f(x)\, dx = 1.$$

Calculate $E(X)$.