MATH 425 11/04/2022

## Continuous random variables

$$P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt$$

$f(t)$ is some function (not necessarily continuous) called the <u>density</u>.

<u>Statistics</u>:

$$E(X) = \int_{-\infty}^{\infty} t f(t)\,dt$$

One has

$$E(g(X)) = \int_{-\infty}^{\infty} g(t) f(t)\, dt$$

a function

In particular, $E(X^2) = \int_{-\infty}^{\infty} t^2 f(t)\, dt$.

One has

$$\mathrm{var}(X) = E(X^2) - (E(X))^2, \quad \sigma(X) = \sqrt{\mathrm{var}(X)}.$$

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Also note that

$$\int_{-\infty}^{\infty} f(t)\, dt = 1$$

<u>Example:</u> A continuous random variable $X$ has density

$$f(x) = \begin{cases} Cx^2 & 1 \leq x \leq 2 \\ 0 & \text{else} \end{cases}$$

where $C$ is some constant.

1. Find $C$
2. Find $E(X)$
3. Find $\mathrm{var}(X)$, $\sigma(X)$.

$$f(x) = \begin{cases} Cx^2 & 1 \le x \le 2 \\ 0 & \text{else.} \end{cases}$$

① $$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_1^2 Cx^2\,dx = C\left[\frac{x^3}{3}\right]_1^2 = C\left(\frac{8}{3} - \frac{1}{3}\right) = C \cdot \frac{7}{3}$$

$$\boxed{C = \frac{3}{7}}$$

$$\boxed{f(x) = \begin{cases} \frac{3}{7}x^2 & 1 \le x \le 2 \\ 0 & \text{else} \end{cases}}$$

② $$E(X) = \int_1^2 x \cdot \frac{3}{7}x^2\,dx = \frac{3}{7}\int_1^2 x^3\,dx = \frac{3}{7}\left[\frac{x^4}{4}\right]_1^2 = \frac{3}{7}\underbrace{\left(4 - \frac{1}{4}\right)}_{\frac{15}{4}}$$

$$= \frac{45}{28}$$

$$E(X^2) = \int_1^2 x^2 \cdot \frac{3}{7} x^2 \, dx = \frac{3}{7} \int_1^2 x^4 \, dx = \frac{3}{7} \left[ \frac{1}{5} x^5 \right]_1^2 =$$

$$= \frac{3}{35}(32-1) = \underline{\underline{\frac{93}{35}}}$$

$$E(X) = \frac{45}{28}$$

$$\operatorname{var}(X) = \frac{93}{35} - \left(\frac{45}{28}\right)^2 \approx \underline{\underline{0.074}}$$

$$\sigma(X) = \sqrt{0.074} \approx \underline{\underline{0.27}}.$$

**Follow-up question:** For the random variable in the previous example, i.e. with density

$$f(x) = \begin{cases} \frac{3}{7}x^2 & 1 \le x \le 2 \\ 0 & \text{else,} \end{cases}$$

calculate $P(X \geq 1.5)$.

same, continuous $>$

**Solution:**

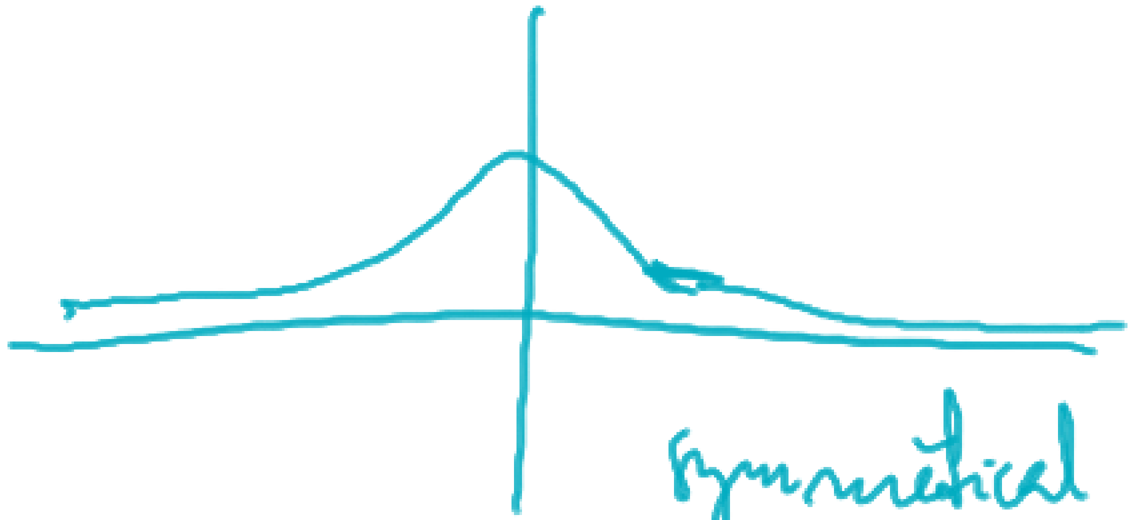
generic approach: $P(X \geq 1.5) = 1 - P(X \leq 1.5)$

$= 1 - \int_{-\infty}^{1.5} f(x)\,dx$

$$\int_{1.5}^{2} \frac{3}{7}x^2\,dx = \frac{3}{7}\left[\frac{x^3}{3}\right]_{3/2}^{2} = \frac{3}{7}\left(\frac{8}{3} - \underbrace{\frac{9}{8}}_{\left(\frac{27}{8}\right)/3}\right) = \frac{3 \cdot 35}{7 \cdot 24} = \underline{\underline{\frac{105}{168}}}$$

**Example:** A continuous random variable has density

$$f(x) = \frac{c}{1+x^2}.$$

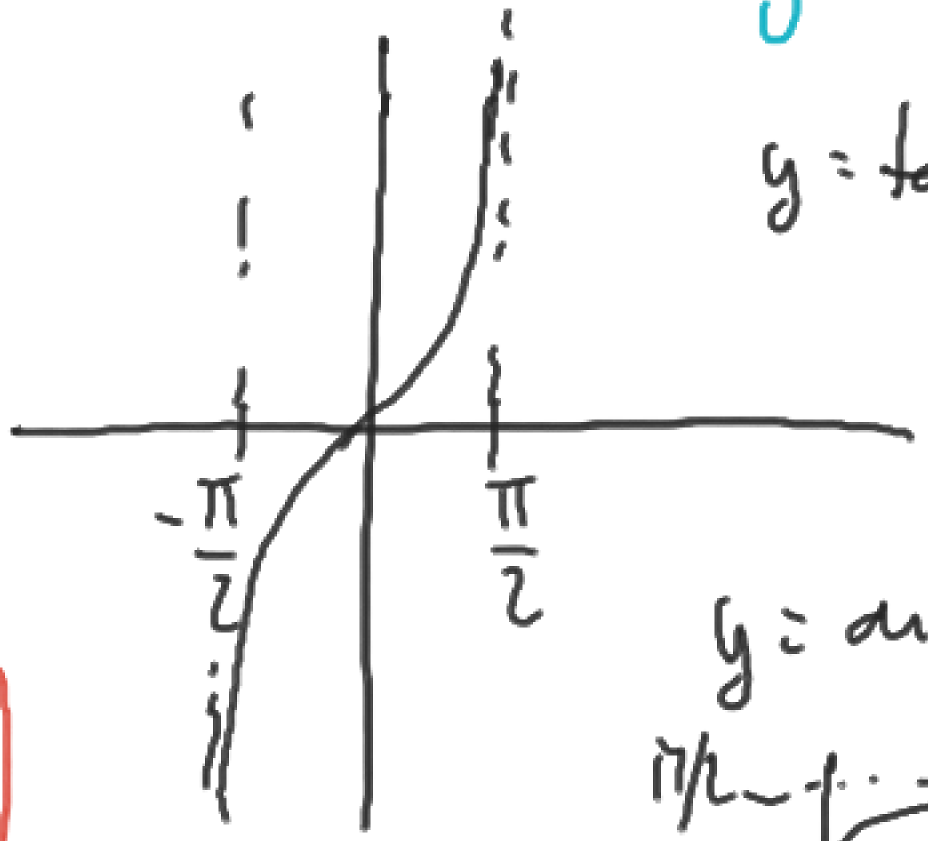


symmetrical (even function)

What is $C$ ?

**Solution:**

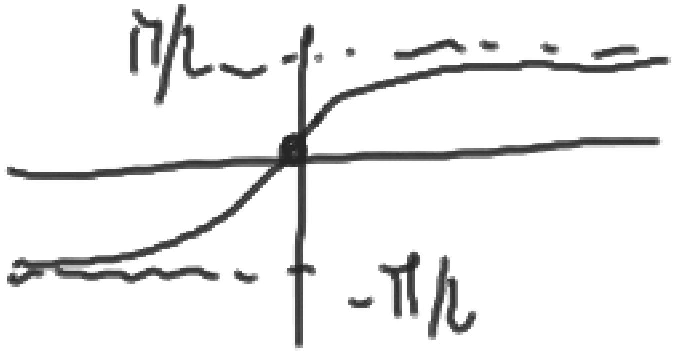
$$1 = C\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = C\left[\arctan(x)\right]_{-\infty}^{\infty}$$




$y = \tan x$

$$= C\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = C\pi$$

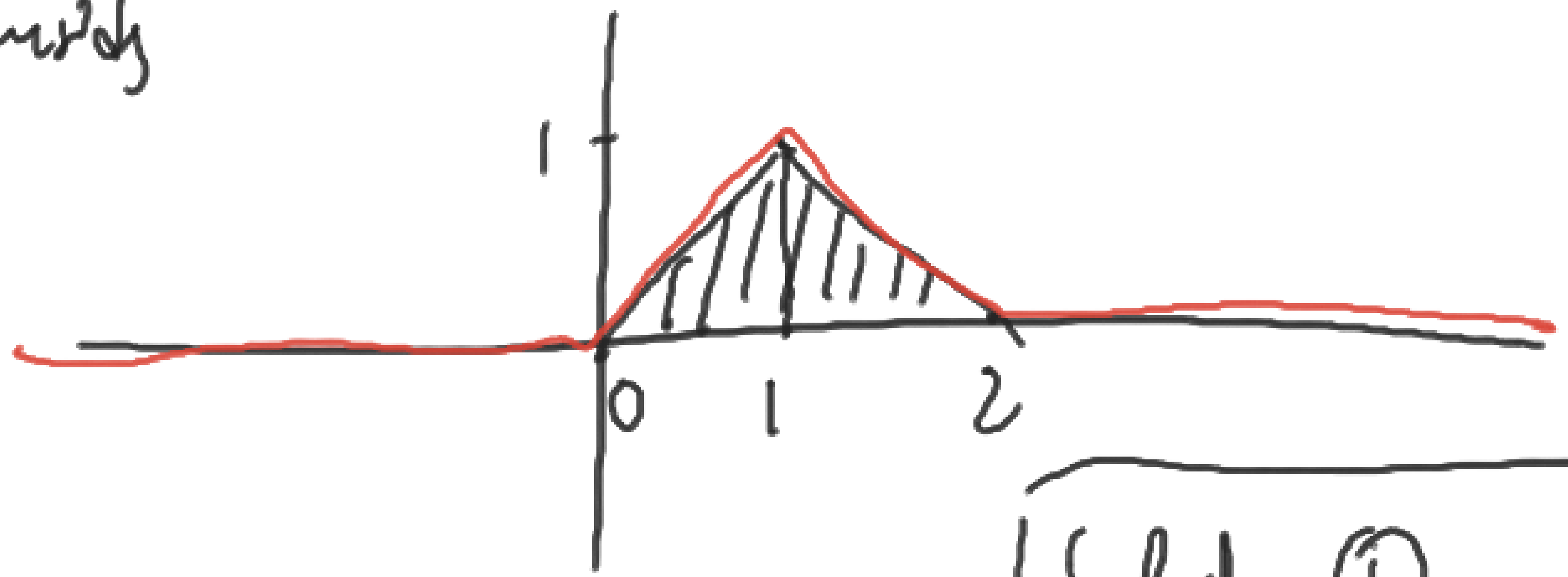
$y = \arctan x$




$$C = \frac{1}{\pi}$$

$$\boxed{\text{density: } f(x) = \frac{1}{\pi(1+x^2)}}$$

Example: $X$ is a continuous random variable with density



$$f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2-x & 1 \le x \le 2 \\ 0 & \text{else} \end{cases}$$

① Calculate $P\left(\frac{1}{2} \le X \le \frac{5}{4}\right)$

② Calculate $E(X)$, $\operatorname{var}(X)$, $\sigma(X)$.

Solution ①:

$$\int_{1/2}^{5/4} f(x)\,dx = \int_{1/2}^{1} x\,dx + \int_{1}^{5/4} (2-x)\,dx$$

$$= \left[\frac{x^2}{2}\right]_{1/2}^{1} + \left[2x - \frac{x^2}{2}\right]_{1}^{5/4} = \frac{1}{2} - \frac{1}{8} + \frac{5}{2} - \frac{25}{32} - 2 + \frac{1}{2} = \underline{\underline{\frac{19}{32}}}$$

$$f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2-x & 1 \le x \le 2 \\ 0 & \text{else} \end{cases}$$


$E(X) = 1$

$$E(X) = \int_0^1 x^2\,dx + \int_1^2 \underbrace{x(2-x)}_{(2x-x^2)}\,dx = \left[\frac{x^3}{3}\right]_0^1 + \left[x^2 - \frac{x^3}{3}\right]_1^2 = \frac{1}{3} + 4 - \frac{8}{3} - 1 + \frac{1}{3} =$$

$$= \underline{\underline{1}}$$

$$E(X^2) = \int_0^1 x^3\,dx + \int_1^2 \underbrace{x^2(2-x)}_{2x^2-x^3}\,dx = \left[\frac{x^4}{4}\right]_0^1 + \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_1^2 =$$

$$= \frac{1}{4} + \frac{16}{3} - 4 - \frac{2}{3} + \frac{1}{4} = \frac{1}{2} + \frac{14}{3} - \frac{12}{3} =$$

$1/2$

$$= \frac{1}{2} + \frac{2}{3} = \frac{7}{6} \,//$$

$$\operatorname{var}(X) = \frac{7}{6} - 1 = \frac{1}{6} \qquad \sigma(X) = \frac{1}{\sqrt{6}}$$

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HW Let $X$ be a continuous random variable with density

$$f(x) = \begin{cases} c & 0 \leq x \leq 1 \\ cx & 1 \leq x \leq 2 \\ 0 & \text{else.} \end{cases}$$

1. Calculate $c$.

2. Calculate $P\left(\frac{2}{3} \leq X \leq \frac{3}{2}\right)$.

3. Calculate $E(X)$

4. Calculate $var(X), \sigma(X)$.