MATH 425 11/7/2022

Example: Suppose a continuous random variable X has density

$$f(x) = \begin{cases} c e^{x} & 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

(a) Find c

(b) Find $E(X)$, $\mathrm{var}(X)$, $\sigma(X)$.

Solution: (a) $1 = c\int_0^1 e^x dx = c\left[e^x\right]_0^1 = c(e-1)$

$$\boxed{c = \frac{1}{e-1}}$$

Density: $f(x) = \frac{1}{e-1} e^x \quad 0 \le x \le 1$

$0 \quad$ else

leibniz rule

$(uv)' = u'v + uv'$

Primitive function of xe^x:

try $(xe^x)' = xe^x + e^x$

$(xe^x - e^x)' = xe^x$

b) $\frac{1}{e-1}\int_0^1 xe^x \, dx = \frac{1}{e-1}\left[xe^x - e^x\right]_0^1$

$= \frac{1}{e-1}\left[e - e - 0 + 1\right] = \frac{1}{e-1}$

$$\boxed{E(X) = \frac{1}{e-1}}$$

function has primitive $2xe^x - 2e^x$

$(x^2e^x)' = x^2e^x + 2xe^x$

$(x^2e^x - 2xe^x + 2e^x)' = x^2e^x$

$E(X^2) = \frac{1}{e-1}\int_0^1 x^2 e^x \, dx = \frac{1}{e-1}\left[x^2e^x - 2xe^x + 2e^x\right]_0^1$

$= \frac{1}{e-1}\left(e - 2e + 2e - 2\right) = \frac{e-2}{e-1}$

$$\boxed{\mathrm{var}(X) = \frac{e-2}{e-1} - \frac{1}{(e-1)^2} = \frac{(e-1)(e-2)-1}{(e-1)^2}}$$

$$\boxed{\sigma(X) = \frac{\sqrt{(e-1)(e-2)-1}}{e-1}}$$

Example: let X be a continuous random variable with density

$$f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{else.} \end{cases}$$

Calculate $E(e^X)$.

Solution:

$$\boxed{E(g(X)) = \int_{-\infty}^{\infty} g(x)\, f(x)\, dx}$$

$$E(e^X) = \int_0^1 e^x\, dx = [e^x]_0^1 = \underline{\underline{e-1}}$$

Transforming the density

Example: Let X be a continuous random variable with density

$$f(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{else.} \end{cases}$$

What is the density of $Y = X^2$?

Solution: $Y = X^2$, same for values: $\boxed{y = x^2}$ Suppose Y has density $g(y)$:

$$\boxed{g(y)\,dy = f(x)\,dx} \rightsquigarrow g(y)\,2x\,\cancel{dx} = f(x)\,\cancel{dx}$$

$$dy = dx^2 = (x^2)'dx = 2x\,dx$$

$$g(y) \cdot 2x = f(x)$$

$$g(y) = \frac{f(x)}{2x} = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

$$df(x) = f'(x)\,dx \quad \left(\frac{df(x)}{dx} = f'(x)\right)$$

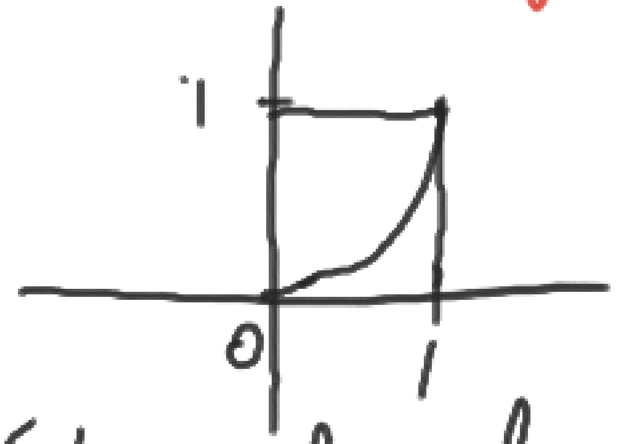
$$g(y) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

express this in terms of y

Answer:

$$\boxed{g(y) = \begin{cases} 1 & 0 \le y \le 1 \\ 0 & \text{else} \end{cases}}$$

$y = x^2$



$0 \le y \le 1$ and each value occurs exactly once

<u>Example</u>: let X be a continuous random variable with density

$$f(x) = \begin{cases} 1 & 1 \le x \le 2 \\ 0 & \text{else.} \end{cases}$$

What is the density $g(y)$ of $Y = X^2$?

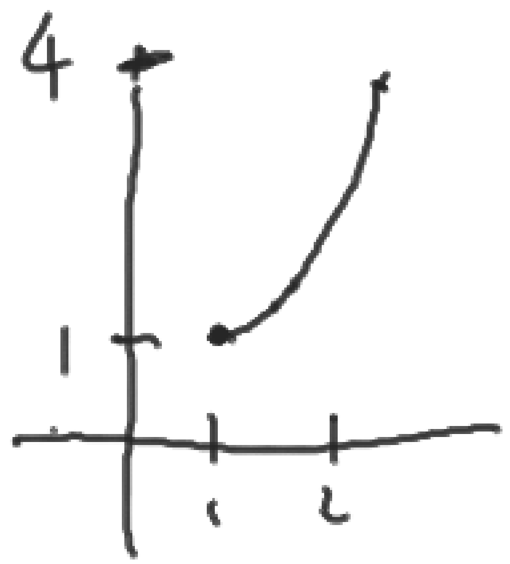
<u>Solution</u>: $\boxed{y = x^2}$ $\qquad \boxed{g(y)\,dy = f(x)\,dx}$ $\qquad dy = dx^2 = 2x\,dx$

$$g(y) \cdot 2x\,\cancel{dx} = f(x)\,\cancel{dx}$$

$$g(y) = \underbrace{\frac{1}{2x}}$$

$$\underbrace{1 \le x \le 2}_{1 \le y \le 4}$$

} Now get rid of x (in favor of y)!



$y = x^2 \qquad x = \sqrt{y}$

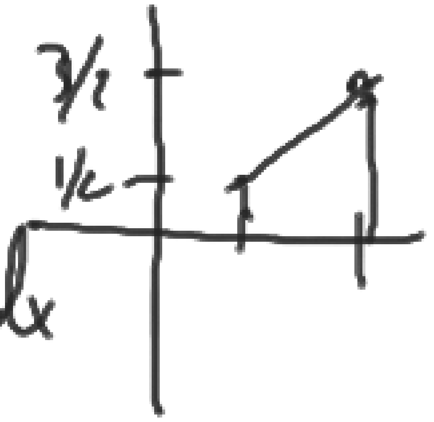
$$\boxed{\textbf{Answer}: \quad g(y) = \begin{cases} \dfrac{1}{2\sqrt{y}} & 1 \le y \le 4 \\ 0 & \text{else} \end{cases}}$$

Example: let $f(x) = \begin{cases} x - \frac{1}{2} & 1 \le x \le 2 \\ 0 & \text{else.} \end{cases}$

check density:

$$\int_1^2 2x - 1\, dx = [x^2 - x]_1^2 = 2$$

let X be a continuous random variable with density $f(x)$. Find the density $g(y)$ of $Y = X^3$.



Solution: $\boxed{y = x^3}$ $\quad \boxed{g(y)\,dy = f(x)\,dx}$ $\quad dy = dx^3 = 3x^2\,dx$

$$g(y)\,3x^2\,\cancel{dx} = \left(x - \frac{1}{2}\right)\cancel{dx} \qquad 1 \le x \le 2$$

$$g(y) = \frac{1}{3x} - \frac{1}{6x^2} \qquad 1 \le x \le 2 \qquad \text{eliminate } x$$

$$1 \le y \le 2^3 = 8$$

Solve for x:

$x = y^{1/3}$

$$\boxed{g(y) = \begin{cases} \dfrac{1}{3y^{1/3}} - \dfrac{1}{6y^{2/3}} & 1 \le y \le 8 \\ 0 & \text{else} \end{cases}}$$

Answer

HW 1 Let X be a continuous random variable with density

$$f(x) = \begin{cases} C \sin x & 0 \le x \le \pi \\ 0 & \text{else} \end{cases}$$

a Find C

b Find $E(X)$, $\mathrm{var}(X)$, $\sigma(X)$.

(2) Let X be a continuous random variable with density

$$f(x) = \begin{cases} 2x-4 & 2 \leq x \leq 3 \\ 0 & \text{else} \end{cases}$$

Find the density $g(y)$ of $Y = \sqrt{X}$.

$$\left[x^2 - 4x\right]_2^3 =$$

$$= 9 - 12 - 4 + 8 = 1$$

✓

$$\int x \sin x \, dx =$$

$u = x \qquad v' = \sin x$

$u' = 1 \qquad v = -\cos x$

$$= uv - \int u'v = -x\cos x - \int -\cos x \, dx =$$

$$= -x\cos x + \sin x + C$$

$$(-x\cos x + \sin x)' =$$

$$= -\cos x + x\sin x + \cos x$$

$$= \underline{\underline{x \sin x}}$$