MATH 425 11/9/2022

<u>Jointly distributed random variables</u> $X_1, \dots X_n$ on the same sample space. ($(X_1, \dots, X_n)$ is sometimes called a random vector). Joint cumulative distribution

$$F(x_1, \dots x_n) = P(X_1 \le x_1, X_2 \le x_2, \dots X_n \le x_n).$$

We call $X_1, \dots X_n$ <u>independent</u> when $X_1 \le x_1, X_2 \le x_2, \dots, X_n \le x_n$ are independent events for any numbers $x_1, x_2, \dots, x_n$.

We say that the random vector $(X_1, \dots, X_n)$ is <u>discrete</u> if it takes on only countably many values.

If (X,Y) is a discrete random vector, we can calculate another statistic

$$\operatorname{cov}(X,Y) = E(X\cdot Y) - E(X)\cdot E(Y)$$

(analogous to the dot product in linear algebra)

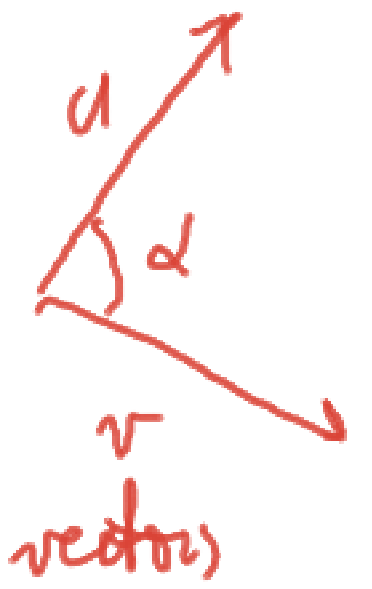
The correlation coefficient:

$$\rho(X,Y) = \frac{\operatorname{cov}(X,Y)}{\sigma(X)\,\sigma(Y)}$$

↖ σ = standard deviation

$$-1 \leq \rho(X,Y) \leq 1$$

analogous to $\cos\alpha = \dfrac{u\cdot v}{\|u\|\,\|v\|}$



$$-1 \leq \cos\alpha \leq 1$$

Note: If $\mathrm{cov}(X,Y)=0$ (equivalently, $\rho(X,Y)=0$), then X,Y are called uncorrelated.

Independent random variables are uncorrelated, but not vice versa

ρ is one number, the independence has more (potentially so many) constraints.

Observe:

$$\mathrm{var}(X+Y) = E\big(\underbrace{(X+Y)^2}_{E(X^2+2XY+Y^2)}\big) - \big(\underbrace{E(X+Y)}_{E(X)+E(Y)}\big)^2 =$$

$$E(X^2+2XY+Y^2) - \underbrace{(E(X)+E(Y))^2}_{(E(X)^2+2E(X)E(Y)+E(Y)^2)}$$

$$= \mathrm{var}(X) + \mathrm{var}(Y) - 2\,\mathrm{cov}(X,Y)$$

$\therefore$ $\boxed{\text{If } X,Y \text{ are uncorrelated then } \mathrm{var}(X+Y) = \mathrm{var}(X) + \mathrm{var}(Y).}$

in particular, independent

analogous to orthogonal in linear algebra: $u \cdot v = 0$

Example (Calculating statistics of jointly distributed discrete random variables): Let X, Y be jointly distributed random variables,

Reading the problem:

$$P(X=0 \,\&\, Y=1) = 0.4$$
$$P(X=1 \,\&\, Y=0) = 0.1$$
$$P(X=-1 \,\&\, Y=-1) = 0.5$$

disjoint events

add up to 1

Calculate $E(X), E(Y), \sigma(X), \sigma(Y), \rho(X,Y)$.

Solution:

$E(X) = 0\cdot 0.4 + 1\cdot 0.1 + (-1)\cdot 0.5$

$= \underline{-0.4}$

$E(X^2) = 0\cdot 0.4 + 1\cdot 0.1 + 1\cdot 0.5 = 0.6$

$\mathrm{var}(X) = E(X^2) - E(X)^2 = 0.6 - 0.16$

$= 0.44$

$\underline{\sigma(X) = \sqrt{0.44}}$

$E(Y) = 1\cdot 0.4 + 0\cdot 0.1 + (-1)\cdot 0.5 = -0.1$ $\quad E(Y^2) = 1\cdot 0.4 + 0\cdot 0.1 + 1\cdot 0.5 = 0.9$

$\mathrm{var}(Y) = 0.9 - 0.01 = 0.89$ $\quad \underline{\sigma(Y) = \sqrt{0.89}}$ $\quad E(XY) = 0\cdot 1\cdot 0.4 + 1\cdot 0\cdot 0.1 + (-1)\cdot(-1)\cdot 0.5 = 0.5$

$\mathrm{cov}(X,Y) = E(XY) - E(X)E(Y) = 0.5 - (-0.4)\cdot(-0.1) = 0.5 - 0.04 = \underline{0.46}$

$$\rho(X,Y) = \frac{0.46}{\sqrt{0.44\cdot 0.89}}$$

Example: let (X,Y) be a random vector

disjoint events $(0,0)$, $(0,1)$, $(-1,1)$ are different pairs

$P(X=0, Y=0) = 0.3$

$P(X=0, Y=1) = 0.2$

$P(X=-1, Y=1) = 0.5$

add up to 1

Calculate $E(X), E(Y), \sigma(X), \sigma(Y), \rho(X,Y)$

Solution: $E(X) = 0\cdot 0.3 + 0\cdot 0.2 + (-1)\cdot 0.5 = \underline{\underline{-0.5}}$ $\quad E(X^2) = 0\cdot 0.3 + 0\cdot 0.2 + 1\cdot 0.5 = 0.5$

$\mathrm{var}(X) = E(X^2) - E(X)^2 = 0.5 - 0.25 = 0.25$ $\quad \sigma(X) = \underline{\underline{0.5}}$

$E(Y) = 0\cdot 0.3 + 1\cdot 0.2 + 1\cdot 0.5 = 0.7$ $\quad E(Y^2) = 0\cdot 0.3 + 1\cdot 0.2 + 1\cdot 0.5 = 0.7$

$\mathrm{var}(Y) = E(Y^2) - E(Y)^2 = 0.7 - 0.49 = 0.21$ $\quad \sigma(Y) = \underline{\underline{\sqrt{0.21}}}$

$E(XY) = 0\cdot 0\cdot 0.3 + 0\cdot 1\cdot 0.2 + (-1)\cdot 1\cdot 0.5 = -0.5$

$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = -0.5 - (-0.5)\cdot 0.7 = -0.5 + 0.35 = -0.15$

$$\boxed{\rho(X,Y) = \frac{-0.15}{0.5\cdot\sqrt{0.21}}}$$

Follow-up note: In neither of those two examples were X,Y independent $(\rho(X,Y) \neq 0)$

HW 3 Suppose (X,Y) is a random vector

$$P(X=2, Y=2) = 0.4$$

$$P(X=1, Y=2) = 0.3$$

$$P(X=1, Y=1) = 0.3.$$

a Calculate $E(X), E(Y), \sigma(X), \sigma(Y)$.

b Calculate $\rho(X,Y)$. Are X, Y independent random variables?

(4) Suppose X, Y are discrete jointly distributed random variables. $E(X) = 0.5$, $E(X^2) = 1$, $E(Y) = 1$, $E(Y^2) = 2$, $\mathrm{var}(X+Y) = 3$. Calculate $\rho(X, Y)$.