MATH 425 11/11/2022

We talked about covariance and correlation coefficient of jointly distributed discrete random variables. We can make the same definitions for continuous jointly distributed random variables:

$$\operatorname{cov}(X,Y) = E(XY) - E(X)E(Y)$$

$$\rho(X,Y) = \frac{\operatorname{cov}(X,Y)}{\sigma(X)\sigma(Y)}.$$

We say that jointly distributed random variables $X_1, \dots, X_n$ are jointly continuous (a continuous random vector $(X_1, \dots, X_n)$) when there exist an n-variable function $f(x_1, \dots, x_n)$ called the joint density such that

$$F(x_1, \dots, x_n) = P(X_1 \le x_1 \;\&\; X_2 \le x_2 \;\&\; \dots \;\&\; X_n \le x_n) =$$

$$= \int_{t_1=-\infty}^{x_1} \dots \int_{t_n=-\infty}^{x_n} f(t_1, \dots, t_n)\, dt_n \dots dt_1$$

What if I just want the density of X_1 (say)?

g

$$g(x_1) = \int_{t_2=-\infty}^{\infty} \cdots \int_{t_n=-\infty}^{\infty} f(t_1, \ldots, t_n)\, dt_n \cdots dt_2$$

This density of X_1 alone is called the <u>marginal density</u>.

What we did on the first slide, in more detail: Let (X,Y) be a continuous random vector with density $f(x,y)$.

Marginal densities:

$$g(x) = \int_{y=-\infty}^{\infty} f(x,y)\,dy \qquad h(y) = \int_{x=-\infty}^{\infty} f(x,y)\,dx$$

marginal density of X

marginal density of Y

$$E(X) = \int_{x=-\infty}^{\infty} x\,g(x)\,dx = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} x\,f(x,y)\,dy\,dx$$

$$E(X^2) = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} x^2 f(x,y)\,dy\,dx$$

$$E(Y) = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} y\,f(x,y)\,dy\,dx$$

$$E(Y^2) = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} y^2 f(x,y)\,dy\,dx$$

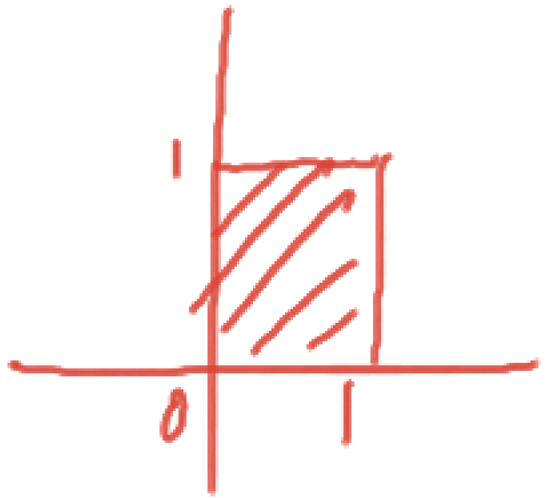
$$E(XY) = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} xy\, f(x,y)\, dy\, dx$$

$$\mathrm{var}(X) = E(X^2) - E(X)^2, \quad \mathrm{var}(Y) = E(Y^2) - E(Y)^2$$

$$\mathrm{cov}(X,Y) = E(XY) - E(X)E(Y) \qquad \rho(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sigma(X)\sigma(Y)}$$

Example: Let X, Y be jointly distributed random variables with joint density

$$f(x,y) = \begin{cases} c(x+y) & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{else} \end{cases}$$



(a) Calculate c.

(b) Calculate the marginal densities of X, Y.

(c) Are X, Y independent?

$\boxed{\text{NO on (c)}}$

(d) Calculate $\rho(X,Y)$.

(?)

If X, Y are independent then the joint density is the product of marginal densities:

$$f(x,y) = g(x)\,h(y)$$

("separation of variables" in ODE (Math 316))

(a)

$$1 = C\int_0^1\int_0^1 (x+y)\,dx\,dy = C\int_0^1 \left[\frac{x^2}{2} + xy\right]_{x=0}^1 dy =$$

$$= C\int_0^1 \left(\frac{1}{2}+y\right)dy = C\left[\frac{1}{2}y + \frac{1}{2}y^2\right]_0^1 = C\left(\frac{1}{2}+\frac{1}{2}\right) = C$$

$$\boxed{C = 1}$$

$$f(x,y) = \begin{cases} x+y & 0 \le x \le 1 \quad 0 \le y \le 1 \\ 0 & \text{else} \end{cases}$$

(b)

marginal density of X

$$g(x) = \int_0^1 (x+y)\,dy = \left[xy + \frac{1}{2}y^2\right]_0^1 = \underline{\underline{x + \frac{1}{2}}}$$

marginal density of Y

$$h(y) = \int_0^1 (x+y)\,dx = \left[xy + \frac{1}{2}x^2\right]_0^1 = \underline{\underline{y + \frac{1}{2}}}$$

(d) $E(X) = \int_0^1 x\underbrace{\left(x+\frac{1}{2}\right)}_{x^2+\frac{x}{2}}dx =$

$g(x) = x + \frac{1}{2} \quad 0 \le x \le 1$

$\quad\quad 0 \quad \text{else}$

$= \left[\frac{x^3}{3} + \frac{x^2}{4}\right]_0^1 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$

$E(X^2) = \int_0^1 \underbrace{x^2\left(x+\frac{1}{2}\right)}_{x^3+\frac{x^2}{2}}dx = \left[\frac{x^4}{4} + \frac{x^3}{6}\right]_0^1 = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$

$var(X) = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{60-49}{144} = \frac{11}{144}$

$\sigma(X) = \frac{\sqrt{11}}{12} = \sigma(Y)$

Y has the same marginal distribution

$\therefore E(Y) = \frac{7}{12}$

$var(Y) = \frac{11}{144}$

$$E(XY) = \int_{x=0}^{1}\int_{y=0}^{1} \underbrace{xy(x+y)}_{x^2y + xy^2}\, dy\, dx =$$

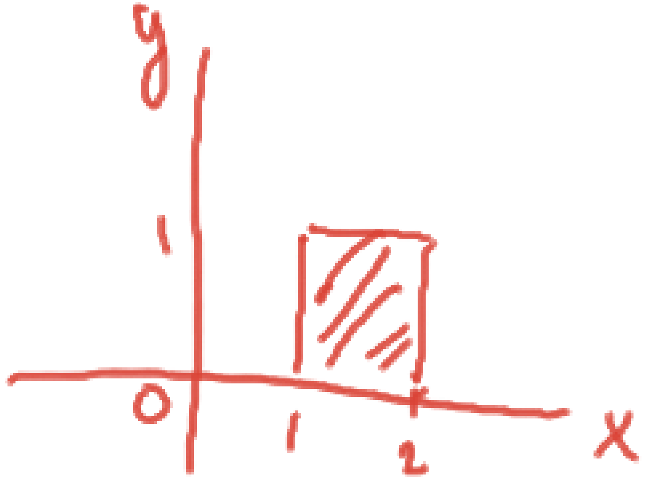
$$= \int_{x=0}^{1} \underbrace{\left[\frac{x^2y^2}{2} + \frac{xy^3}{3}\right]_{y=0}^{1}}_{\frac{x^2}{2} + \frac{x}{3}} dx = \int_0^1 \left(\frac{x^2}{2} + \frac{x}{3}\right) dx = \left[\frac{x^3}{6} + \frac{x^2}{6}\right]_0^1 = \frac{1}{3}$$

$$\operatorname{cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{3} - \frac{49}{144} = \frac{-1}{144}$$

$$\rho(X,Y) = \frac{-1/144}{(\sqrt{11}/12)^2} = -\frac{1}{11}$$

HW 5 Let (X,Y) be a continuous random vector with joint density

$$f(x,y) = \begin{cases} c(x-y) & 1 \leq x \leq 2,\ 0 \leq y \leq 1 \\ 0 & \text{else} \end{cases}$$



(a) Calculate c

(b) Calculate the marginal densities $g(x)$ of X and $h(y)$ of Y

(c) Are X, Y independent?

(d) Calculate $E(X), E(Y), \sigma(X), \sigma(Y)$

(e) Calculate $\rho(X,Y)$.