MATH 425 11/14/2022

Change of variables ← substitution in multiple (at least let's mention double) integrals (or densities).

<u>Example</u>: Suppose X, Y are jointly continuous jointly distributed random variables with joint density $f(x,y)$. Find the density $h(z)$ of $Z = X + Y$.

<u>Solution</u>: change variables from (X,Y) to (X,Z). Transform the density and then take the marginal density for Z.

$$f(x,y)\,dx\,dy = g(\overset{t}{x},z)\,d\overset{t}{x}\,dz$$

$$z = x+y$$

$$t = x$$

Jacobian: $\dfrac{\partial t}{\partial x}$ $\dfrac{\partial t}{\partial y}$

$$\underset{dt}{dx}\,dz = \det\begin{pmatrix} \dfrac{\partial x}{\partial x} & \dfrac{\partial x}{\partial y} \\ \dfrac{\partial z}{\partial x} & \dfrac{\partial z}{\partial y} \end{pmatrix} dx\,dy$$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\det\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = 1 - 0 = 1$$

In the case of our example,

$$dx\,dy = dx\,dz.$$

$$f(x,y) = g(x,z)$$

$$z = x+y$$

$$y = z - x$$

The joint density of (X,Z) is $g(x,z) = f(x, z-x)$.

Marginal density for Z : $\displaystyle\int_{x=-\infty}^{\infty} g(x,z)\,dx = \boxed{\int_{x=-\infty}^{\infty} f(x,z-x)\,dx}$

Summary: If X, Y are jointly continuous random variables with density $f(x,y)$, then the density of $Z = X+Y$ is

$$h(z) = \int_{-\infty}^{\infty} f(x, z-x)\,dx.$$

If X, Y are independent with densities $f_1(x)$, $f_2(y)$, then the joint density is $f(x,y) = f_1(x) f_2(y)$.

So if X, Y are independent continuous random variables with densities $f_1(x)$, $f_2(y)$ then the density $h(z)$ of $Z = X+Y$ is:

$$h(z) = \underbrace{\int_{-\infty}^{\infty} f_1(x) f_2(z-x)\,dx}$$

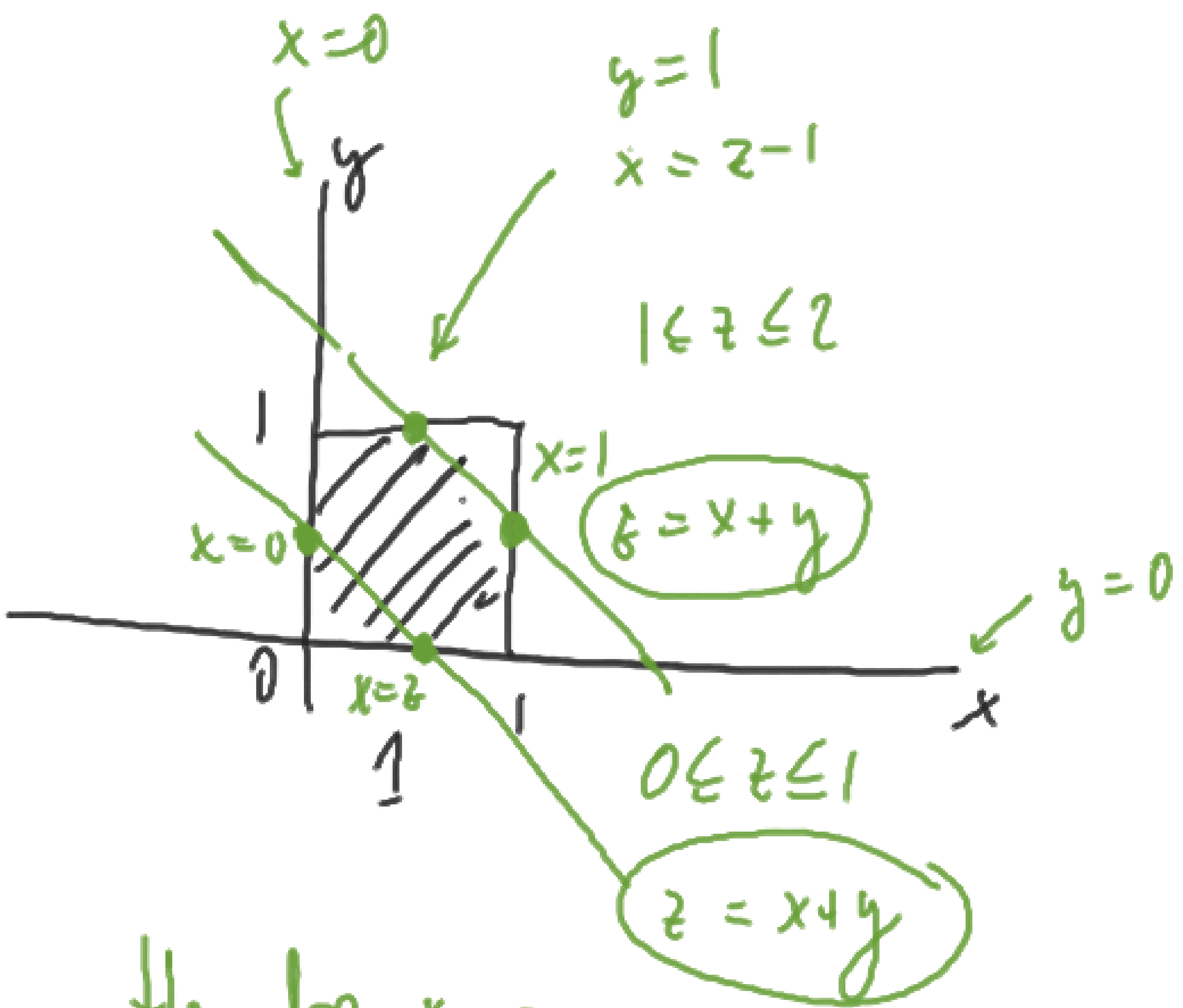
This is known as the <u>convolution</u> $f_1 * f_2\ (z)$.

<u>Example</u>: let X, Y be independent continuous random variables which both have the same density

$$f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$$

(the uniform distribution on $[0,1]$).

Calculate the density $h(z)$ of $X+Y$.



Solution:

$0 \le z \le 1$

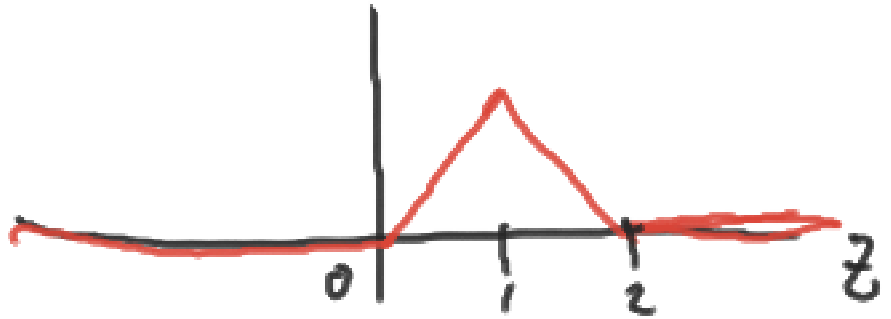
$$\int_{x=0}^{z} 1\cdot 1\,dx = [x]_0^z = z$$

$1 \le z \le 2$

$$\int_{x=z-1}^{1} 1\cdot 1\,dx = [x]_{z-1}^{1} = 2-z.$$

How does x vary

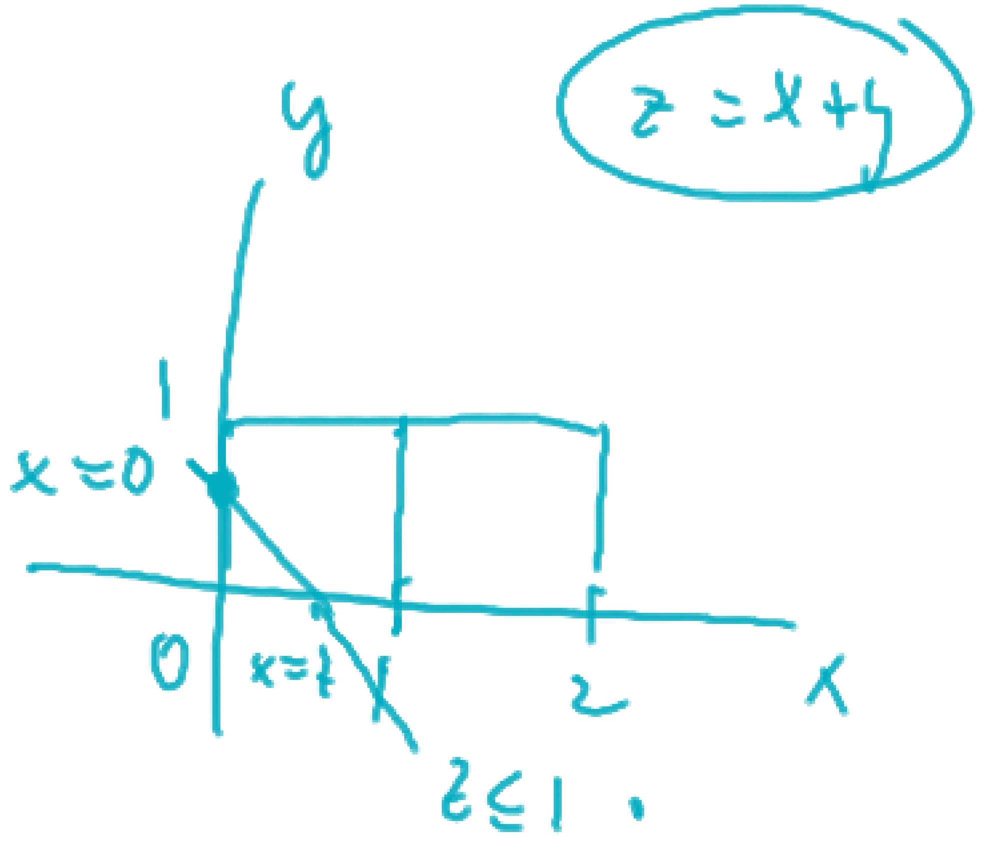
$$\boxed{\text{Answer: } h(z) = \begin{cases} z & 0 \le z \le 1 \\ 2-z & 1 \le z \le 2 \\ 0 & \text{else} \end{cases}}$$



Example: Calculate the density of a sum of three independent random variables, each uniformly distributed on $[0,1]$.

Solution: Start with the previous example density:

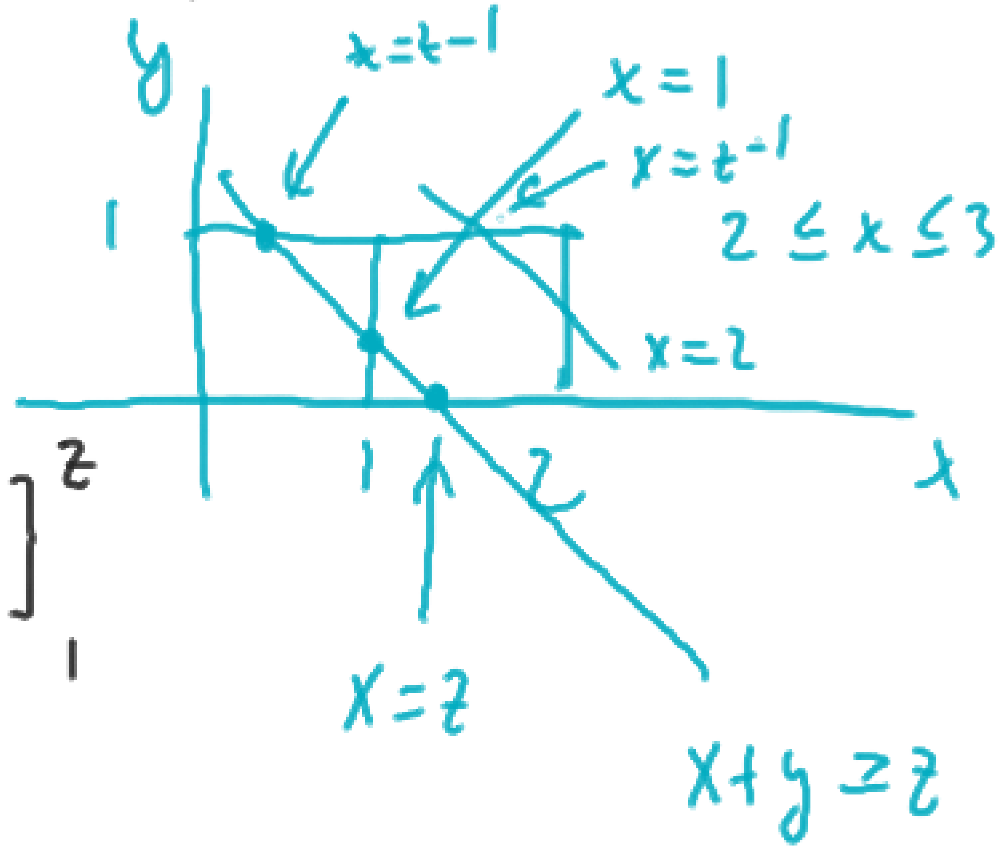
$$f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2-x & 1 \le x \le 2 \\ 0 & \text{else} \end{cases} \qquad g(y) = \begin{cases} 1 & 0 \le y \le 1 \\ 0 & \text{else.} \end{cases}$$



We need the convolution $f * g(z) = \int_{-\infty}^{\infty} f(x)\, g(z-x)\, dx$

$$0 \le z \le 1 \quad = \int_{x=0}^{z} x \cdot 1 \, dx = \left[\frac{x^2}{2}\right]_0^z = \frac{z^2}{2}$$

$$f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2-x & 1 \le x \le 2 \\ 0 & \text{else} \end{cases} \qquad g(y) = \begin{cases} 1 & 0 < y \le 1 \\ 0 & \text{else} \end{cases}$$



$1 \le z \le 2$:

$$\int_{x=z-1}^{1} x \cdot 1\,dx + \int_{1}^{z} (2-x)\cdot 1\,dx = \left[\frac{x^2}{2}\right]_{z-1}^{1} + \left[2x - \frac{x^2}{2}\right]_{1}^{z}$$

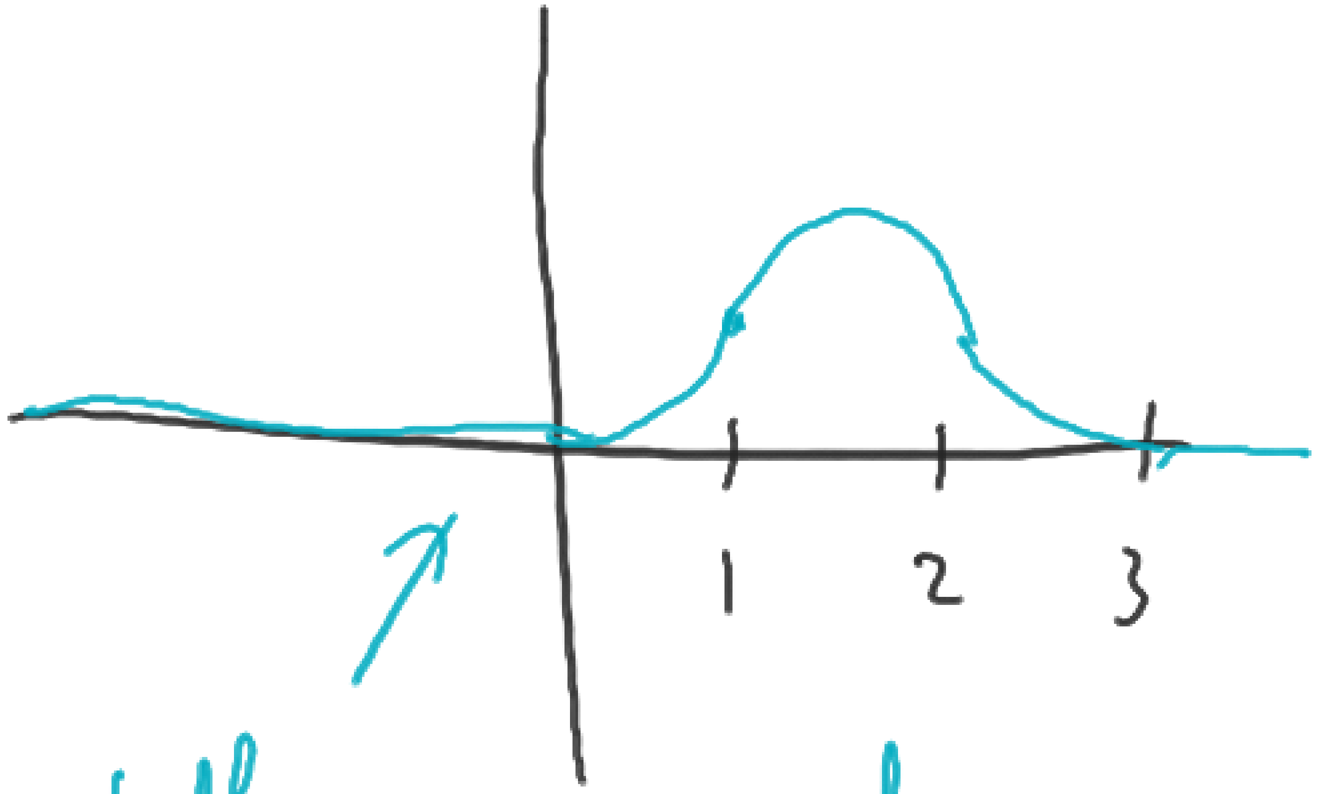
$$= \underbrace{\frac{1}{2} - \frac{(z-1)^2}{2}}_{-\frac{z^2}{2}+z-\frac{1}{2}} + 2z - \frac{z^2}{2} - 2 + \frac{1}{2} = \underline{\underline{-z^2 + 3z - \frac{3}{2}}}$$

$2 \le z \le 3$:

$$\int_{x=z-1}^{2} (2-x)\,dx = \left[2x - \frac{x^2}{2}\right]_{z-1}^{2} = 4 - 2 - 2(z-1) + \frac{(z-1)^2}{2} =$$

$$= 2 - 2z + 2 + \frac{z^2}{2} - z + \frac{1}{2} = \underline{\frac{9}{2} - 3z + \frac{z^2}{2}}$$

$$h(z) = \begin{cases} \frac{z^2}{2} & 0 \le z \le 1 \\ -z^2 + 3z - \frac{3}{2} & 1 \le z \le 2 \\ \frac{9}{2} - 3z + \frac{z^2}{2} & 2 \le z \le 3 \end{cases}$$



better approximation of the bell curve

HW (1) Suppose X, Y are independent continuous random variables.

X has density $f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{else.} \end{cases}$

Y has density $g(y) = \begin{cases} \frac{1}{2} & 0 \le y \le 2 \\ 0 & \text{else.} \end{cases}$

Calculate the density of $Z = X + Y$.