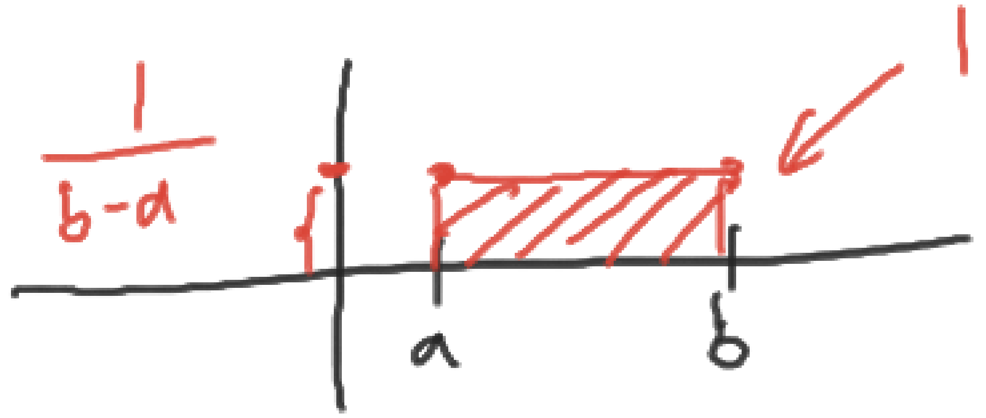
MATH 425 11/16/2022

Examples of continuous distributions

(1) The uniform distribution $U_{a,b}$ has constant density on an interval $[a,b]$. $-\infty < a < b < \infty$. The density is

$$f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{else} \end{cases}$$



Calculating the statistics:

$$E(U_{a,b}) = \frac{a+b}{2}$$

$$\frac{1}{b-a}\int_a^b x\,dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{b^2-a^2}{b-a}\cdot\frac{1}{2} = \frac{b+a}{2}$$

$$E\left((U_{a,b})^2\right) = \frac{1}{b-a}\int_a^b x^2\,dx = \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b = \frac{b^3-a^3}{b-a}\cdot\frac{1}{3} = \frac{1}{3}(b^2+ab+a^2)$$

$$\operatorname{var}(U_{a,b}) = E\left((U_{a,b})^2\right) - E(U_{a,b})^2 = \frac{1}{3}(b^2+ab+a^2) - \frac{1}{4}(a+b)^2 =$$

$$\frac{4b^2+4ab+4a^2-3a^2-6ab-3b^2}{12}$$

$$= \frac{a^2-2ab+b^2}{12} = \frac{(a-b)^2}{12}$$

$$= \frac{(a-b)^2}{12}$$

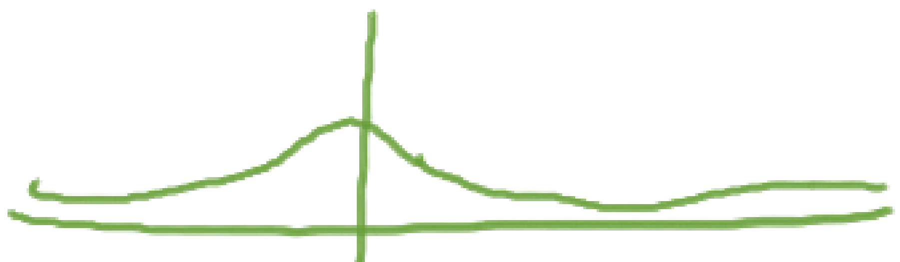
$$\boxed{\sigma(U_{a,b}) = \frac{b-a}{\sqrt{12}}}$$

The normal (Gaussian) distribution

Adding many copies of equally distributed independent random variables with the same E, σ, and scaling to make $E=0, \sigma=1$ always gives the standard normal distribution.

↖ the central limit theorem

Density has to be $f(x) = C e^{-x^2/2}$



an even function

What is the constant:

$$\int_{-\infty}^{\infty} e^{-x^2/2}\, dx$$

Gauss's idea:

we can do this integral in polar coordinates

$$\left(\int_{-\infty}^{\infty} e^{-x^2/2}\,dx\right)^2 = \left(\int_{-\infty}^{\infty} e^{-x^2/2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2/2}\,dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\,dx\,dy$$

$$\boxed{dx\,dy = r\,dr\,d\theta}$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$dx\,dy = \underbrace{\det\begin{pmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{pmatrix}}\,dr\,d\theta$$

$$\det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

$$= \int_{r=0}^{\infty}\int_{\theta=0}^{2\pi} r e^{-r^2/2}\,d\theta\,dr = 2\pi\int_{r=0}^{\infty} r e^{-r^2/2}\,dr$$

$$u = \frac{r^2}{2}$$

$$du = r\,dr$$

$$= 2\pi\int_{u=0}^{\infty} e^{-u}\,du = 2\pi\left[-e^{-u}\right]_0^{\infty} = 2\pi\left(0-(-1)\right) = 2\pi$$

$$\boxed{\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}}$$

$$C = \frac{1}{\sqrt{2\pi}}$$

The density of the standard normal distribution is

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}}$$

A standard normal variable is often denoted by Z.

$$E(Z) = 0$$

$$\mathrm{var}(Z) = E(Z^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2}\, dx$$

because $E(Z)=0$

$$= \frac{1}{\sqrt{2\pi}} \left(\underbrace{\left[-x e^{-x^2/2} \right]_{-\infty}^{\infty}}_{0} + \underbrace{\int_{-\infty}^{\infty} e^{-x^2/2}\, dx}_{\sqrt{2\pi}} \right)$$

$$= 1$$

$$\left(-x e^{-x^2/2} \right)' = x^2 e^{-x^2/2} - e^{-x^2/2}$$

$$\boxed{E(Z) = 0 \qquad \mathrm{var}(Z) = 1 \qquad \sigma(Z) = 1}$$

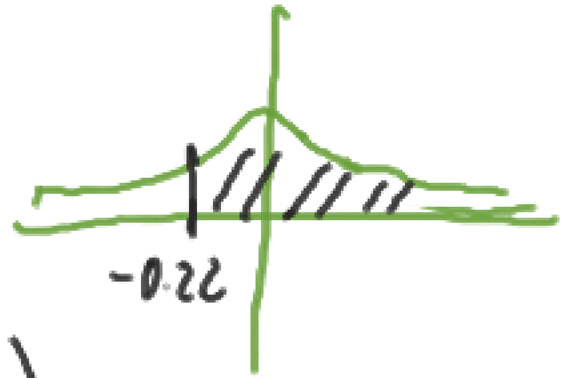
The <u>Z-table</u> (on the course webpage) is a table of the cumulative distribution of Z:

$$P(x) = \int_{-\infty}^{x} e^{-t^2/2}\,dt$$

<u>Example</u>: (a) $P(Z < 0.49) \doteq \underline{\underline{0.6879}}$

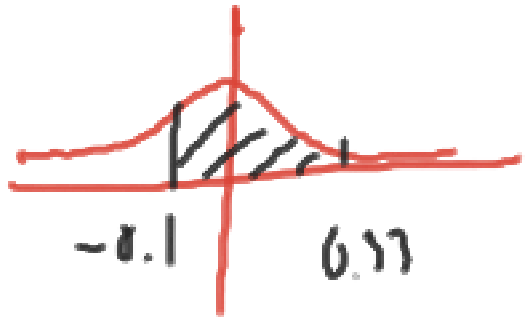
(b) $P(Z > 0.31) = 1 - P(Z \leq 0.31) \doteq 1 - 0.6217 = \underline{\underline{0.3783}}$

(c) $P(Z > -0.22) = P(Z < 0.22) \doteq 0.5871$



(d) $P(-0.1 < Z < 0.33) = P(Z < 0.33) - P(Z < -0.1)$

$= P(Z < 0.33) - P(Z > 0.1) =$



$= P(Z < 0.33) - 1 + P(Z < 0.1) = 0.6293 - 1 + 0.5398 = \underline{\underline{0.1691}}$

MW 2 Let Z be a random variable with the standard normal distribution. Calculate

a) $P(Z < 3.25)$

b) $P(Z > 0.56)$

c) $P(Z > -0.44)$

d) $P(-0.18 < Z < 1.57)$

Use the Z table from the course webpage.