MATH 425 11/18/2022

Recall the standard normal distribution Z with density

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}.$$

We have $E(Z)=0$, $\sigma(Z)=1$. Can we make a variable $Z_{\mu,\sigma}$ which is normally distributed but have $E(Z_{\mu,\sigma}) = \mu$, $\sigma(Z_{\mu,\sigma}) = \sigma$?
All we need to do is to define

$$Z_{\mu,\sigma} = \sigma Z + \mu.$$

(scaling scales the standard deviation linearly, shifting does not affect it but shifts the expectation).

Calculating the density of $X = Z_{\mu,\sigma}$:

$$x = \sigma z + \mu$$

$$dx = \sigma dz$$

$$z = \frac{x-\mu}{\sigma}$$

$$g(x) \cdot \sigma = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

$$g(x)\,dx = f(z)\,dz$$

$$g(x)\,dx = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$g(x) = \frac{1}{\sqrt{2\pi} \cdot \sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^2/2}$$

$$\boxed{\text{The density of } Z_{\mu,\sigma} \text{ is } \frac{1}{\sqrt{2\pi} \cdot \sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^2/2}.}$$

$$Z_{\mu,\sigma} = \sigma Z + \mu \quad , \quad Z = \frac{Z_{\mu,\sigma} - \mu}{\sigma}$$

<u>Example</u>: Calculate $P(0.5 < Z_{1,2} < 1.5)$

<u>Solution</u>: $\mu = 1$, $\sigma = 2$

$$Z = \frac{Z_{\mu\sigma} - \mu}{\sigma} \qquad Z = \frac{Z_{1,2} - 1}{2}$$

$$P\left(\frac{0.5-1}{2} < Z < \frac{1.5-1}{2}\right)$$

$$P(-0.25 < Z < 0.25) = P(Z < 0.25) - P(Z < -0.25)$$

$$= P(Z < 0.25) - 1 + P(Z > -0.25) = 2P(Z < 0.25) - 1 =$$

$$= 2 \cdot 0.5987 - 1 = \underline{\underline{0.1974}}$$

Example: Mr. Z is due at work by 9 AM. His trip to work takes on average 45 min since the time he wakes up, with a standard deviation of 15 min. If he wakes up at 8 AM, what is the probability that he makes it to work on time?

Solution: We assume the length of the trip has a normal distribution. Counting in minutes, the length of Mr. Z's trip has the distribution of

$$Z_{45,15}$$

We are calculating

$$P(Z_{45,15} < 60) = P\left(Z < \frac{60-45}{15}\right)$$

$$Z = \frac{Z_{45,15} - 45}{15} \qquad = P(Z < 1) \doteq \underline{\underline{0{,}8413}}$$

<u>Fact</u>: If X, Y are independent random variables distributed as Z_{μ_1,σ_1} and Z_{μ_2,σ_2}, then $X+Y$ is distributed as

$$Z_{\mu_1+\mu_2, \sqrt{\sigma_1^2+\sigma_2^2}} .$$

The key point is that a sum of independent normal variables is normal.

(Proof by calculating the convolution

$$\frac{1}{\sigma_1\sqrt{2\pi}} e^{-\left(\frac{x-\mu_1}{\sigma_1}\right)^2/2} * \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\left(\frac{x-\mu_2}{\sigma_2}\right)^2/2}.$$

Hint: perfect square.)

This motivates the Central limit theorem:

Let $X_1, X_2, X_3, \dots$ be independent random variables which are equally distributed and $E(X_i) = \mu$, $\sigma(X_i) = \sigma$.

Then
$$\lim_{n\to\infty} \left(\frac{(X_1 + \cdots + X_n) - n\mu}{\sqrt{n}\,\sigma} \right)$$

$E(X_1 + \cdots + X_n) = n\mu$

$\sigma(X_1 + \cdots + X_n) = \sqrt{n}\,\sigma.$

$E = 0,\ \sigma = 1.$

exists with probability 1 and has the standard normal distribution.

This applies to the binomial, negative binomial, Poisson).

Example: A fair coin is flipped 100 times.
What is the probability that H will come up at most 40 times?

Solution: Approximate by the normal distribution.

$X = \# H$

$E(X) = 50$

$\sigma(X) = 5$

40 ✓ | 41 ✗

discrete variable

$\mathrm{var}(X_{n,p}) = np(1-p)$

$\sigma(X_{n,p}) = \sqrt{n}\sqrt{p(1-p)}$

$Z \approx \dfrac{X-50}{5}$

$P\left(Z < \dfrac{40.5-50}{5}\right) = P(Z < -1.9) = 1 - P(Z < 1.9)$
$\doteq 1 - 0.9713 = 0.0287$

HW 3 Flip the fair coin 100 times. Use the normal approximation to calculate the probability that it comes up <u>exactly</u> 50 times. (use $49.5 < Z_{\mu,\sigma} < 50.5$)

4 Mr. Z takes 45 minutes to get to work with standard deviation of 15 minutes. He is due at work at 9AM. Today he overslept and got up at 8:00. What is the probability he still makes it to work on time?
(no halves here, time is continuous)