MATH 425 11/28/2022

Exponential distribution = negative $\underbrace{\text{Poisson distribution}}$ $\leftarrow Y_\lambda$
with $r=1$.

the number of people who will call a center in a given unit of time, if the average number of callers is $\lambda$.

$E_\lambda$ = what amount of time it takes before the first caller calls in?

For any distribution $F$, the <u>Hazard rate</u> is the conditional density under the assumption that $X \geq t$.

$$\lambda(t) = \frac{f(t)}{1-F(t)} \qquad f(t) = F'(t).$$

For the exponential distribution, the hazard rate is constant.

$$\lambda(t) = \lambda.$$

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This leads to a differential equation

$$\frac{F'(t)}{1-F(t)} = \lambda$$

$$\frac{\frac{dy}{dt}}{1-y} = \lambda$$

$$\frac{dy}{1-y} = \lambda\, dt$$

$$-\ln(1-y) = \lambda t + C$$

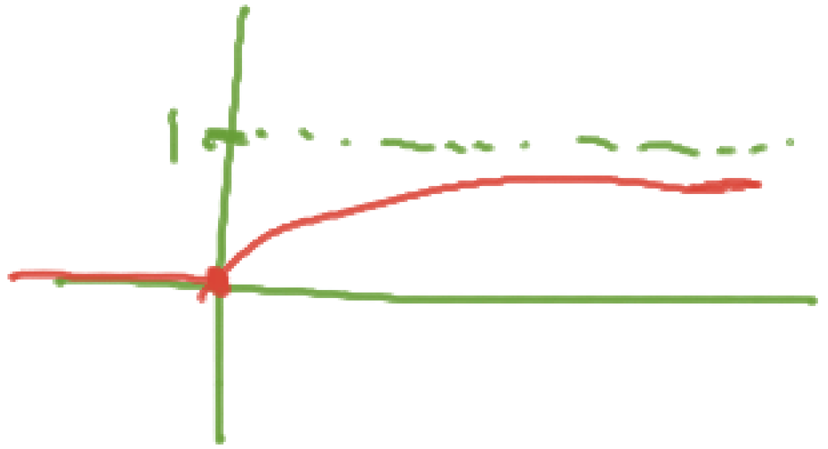
$$-\ln(1-y) = \lambda t + C$$
$$\ln(1-y) = -\lambda t - C$$
$$1-y = K\exp(-\lambda t)$$
$$F(t) = y = 1 - K\exp(-\lambda t)$$

if we measure from time 0 then we want $F(0) = 0$

Then $K = 1$

<u>Answer</u>: The cumulative distribution of an exponential random variable $E_\lambda$ with hazard rate $\lambda$ is

$$F(t) = \begin{cases} 1-\exp(-\lambda t) & t \geq 0 \\ 0 & t \leq 0 \end{cases}$$




The density is

$$f(t) = F'(t) = \begin{cases} \lambda\exp(-\lambda t) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Statistics of an exponentially distributed variable:

$E(E_\lambda) = \frac{1}{\lambda}$ $\quad$ $var(E_\lambda) = \frac{1}{\lambda^2}$ , $\sigma(E_\lambda) = \frac{1}{\lambda}$.

$$E(E_\lambda^2) = \int_0^\infty \lambda t^2 e^{-\lambda t}\, dt =$$

$$= \left[ \lambda t^2 e^{-\lambda t} - 2t e^{-\lambda t} - \frac{2}{\lambda^2} e^{-\lambda t} \right]_0^\infty$$

$$= \frac{2}{\lambda^2}$$

$$var(E_\lambda) = E(E_\lambda^2) - E(E_\lambda)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

$$\left(-t^2 e^{-\lambda t}\right)' = \lambda t^2 e^{-\lambda t} - 2t e^{-\lambda t}$$

$$\left(-\frac{2}{\lambda} t e^{-\lambda t}\right)' = 2t e^{-\lambda t} - \frac{2}{\lambda} e^{-\lambda t}$$

$$\left(-\frac{2}{\lambda^2} e^{-\lambda t}\right)' = \frac{2}{\lambda} e^{-\lambda t}$$

Example: An average of 10 people call a center in an hour. What is the probability that nobody has called during the first 45 min?

Solution via the exp. distribution: $P\left(E_{10} > \frac{3}{4}\right) = 1 - \left(1 - e^{-10 \cdot \frac{3}{4}}\right)$

$$= \underline{\underline{e^{-7.5}}}$$

Solution via the Poisson distribution: The average number of callers in 45 min is $\frac{3}{4} \cdot 10 = 7.5$.

$$P(Y_{7.5} < 1) = e^{-7.5}$$

$$\underbrace{\phantom{Y_{7.5}}}_{=0}$$

$P(Y_\lambda = k) = \frac{\lambda^k}{k!} e^{-\lambda}$

$$\boxed{P(E_\lambda > t) = P(Y_{\lambda t} < 1)}$$

The Poisson process - independent events happening at every moment of time with constant probability rate.

generally, event happening at every moment of time with a probability rate = A continuous stochastic process.

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Example: Half-life

We have a radioactive substance. Half-life = amount of time it takes for half of the substance to decay. How does the half-life relate to the rate of decay $\lambda$ ?

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Solution: 50 percentile of $E_\lambda$.

$F(t) = \frac{1}{2}$ $\quad 1 - e^{-\lambda t} = \frac{1}{2} \Big\}$ $e^{-\lambda t} = \frac{1}{2}$, $e^{\lambda t} = 2$, $\lambda t = \ln 2$, $\boxed{t = \frac{\ln 2}{\lambda}}$

<u>Example</u>: U 235 has half-life $700 \cdot 10^6$ years.

Suppose a rock contains U235, what fraction of the amount of U 235 will be left in the rock after $100 \cdot 10^6$ years?

<u>Solution</u>: $h = 700 \cdot 10^6$ $\quad \lambda = \dfrac{\ln 2}{700 \cdot 10^6}$

$\dfrac{100}{700} = \dfrac{1}{7}$

$$1 - F(100 \cdot 10^6) = 1 - \left(1 - e^{-\lambda 100 \cdot 10^6}\right) = e^{-\frac{\ln 2}{7}} = \left(\frac{1}{2}\right)^{1/7}.$$

In general, at time $h \cdot n$, there will be $\left(\frac{1}{2}\right)^n$ of the substance left (which is why it is called the exponential distribution).

HW

1. If the average number of callers to a call center in an hour is 30, what is the probability that the first caller will call between 30 and 45 minutes from when we started measuring?

2. Suppose a radioactive substance has half-life 100 years. If a rock contains 1 kg of the substance, how much of the substance will it contain in 35 years?