MATH 425 11/30/2022

The negative Poisson distribution of Y_λ with general $r \geq 1$.

($\rightsquigarrow r \in (0, \infty)$), ?!

The Gamma distribution $\Gamma_{r,\lambda}$

Scenario: When will the r-th customer call the center?

("The 9th caller gets a prize.")

$r=1$ we have the exponential distribution. Density

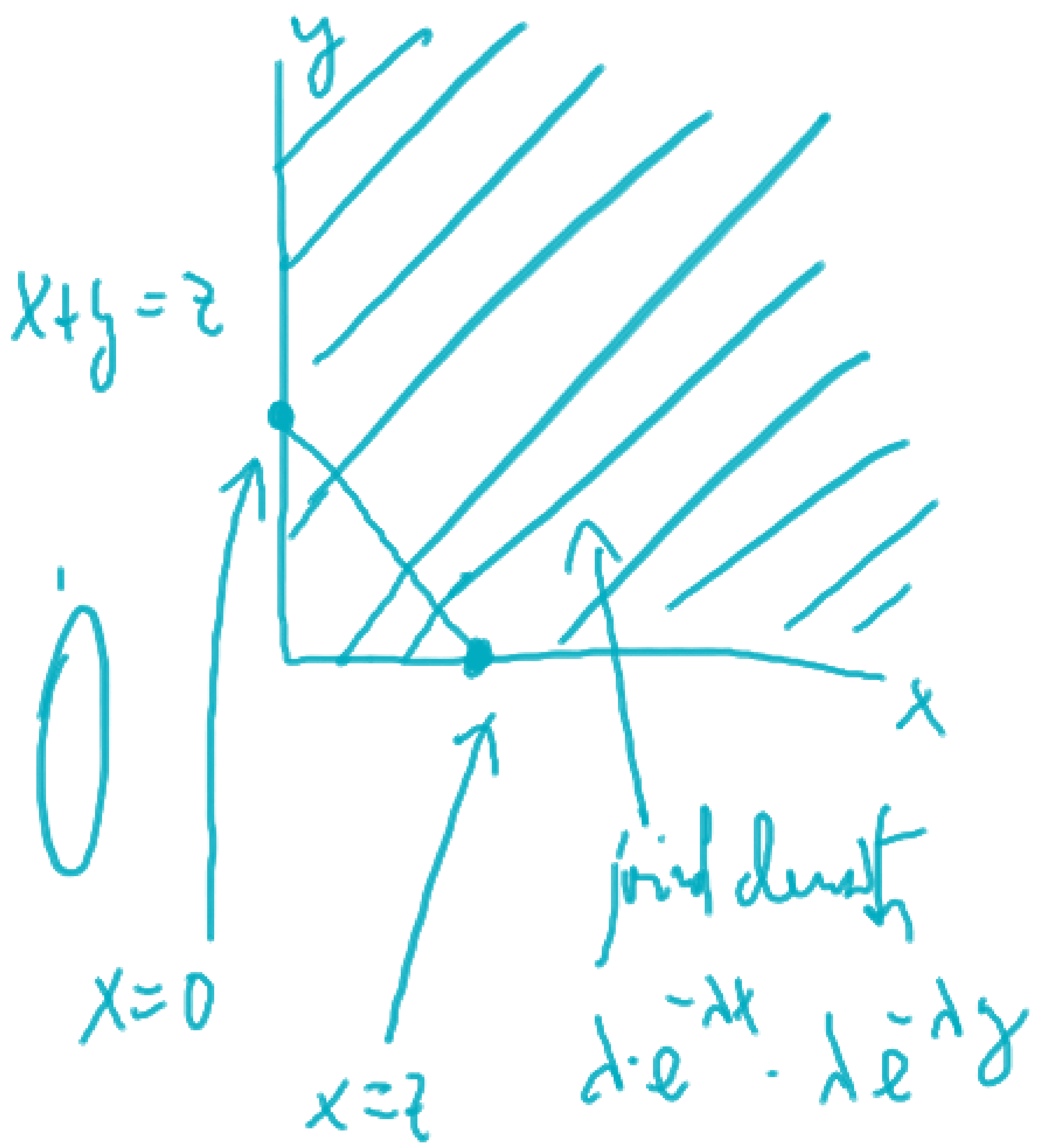
$$f(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & \text{else} \end{cases}$$

For $r = 2$:

constant of integration

$$\underbrace{f * f}_{\text{convolution}}(z) = \int_0^{z} \lambda e^{-\lambda x} \cdot \lambda e^{-\lambda (z-x)} \, dx$$

$$= \lambda^2 e^{-\lambda z} \int_0^{z} dx = \lambda^2 e^{-\lambda z} z$$



$\Gamma_{2,\lambda}$ has density

$$g(z) = \begin{cases} \lambda^2 e^{-\lambda z} z & z \geq 0 \\ 0 & \text{else.} \end{cases}$$

The distribution $\Gamma_{r,\lambda}$ has density

$$g_r(t) = \begin{cases} \dfrac{\lambda^r e^{-\lambda t} t^{r-1}}{(r-1)!} & t \geq 0 \\ 0 & \text{else.} \end{cases}$$

$$E(\Gamma_{r,\lambda}) = \frac{r}{\lambda}$$

$$\operatorname{var}(\Gamma_{r,\lambda}) = \frac{r}{\lambda^2}$$

$$\sigma(\Gamma_{r,\lambda}) = \frac{\sqrt{r}}{\lambda^2}$$

Recall: $E(E_\lambda) = \dfrac{1}{\lambda}$ $\quad \operatorname{var}(E_\lambda) = \dfrac{1}{\lambda^2}$;

Example: Suppose a radio station gets an average of 10 callers per hour. What is the distribution of the time (in minutes) the 9th caller will call in?

Solution: Counting in minutes, $\frac{10}{60} = \frac{1}{6}$ callers per minute.

Answer: $\Gamma_{9,1/6}$.

Follow-up: Approximate by the normal distribution: $E(\Gamma_{9,1/6}) = 54$

$$\sigma(\Gamma_{9,1/6}) = \frac{\sqrt{9}}{1/6} = 18$$

$$Z_{54,18} \qquad \frac{\Gamma_{9,1/6} - 54}{18} \sim Z$$

We can calculate the cumulative distribution of $\Gamma_{r,\lambda}$, $r \in \mathbb{N}_0$ by integrating by parts. But you only uncover the Poisson distribution:

$$P(\Gamma_{r,\lambda} \leq z) = P(Y_{\lambda z} \geqslant r)$$

has to be non-strict (the Poisson distribution is discrete)

an average of λ callers call in a unit of time

what is the probability that at least r callers call in z units of time.

The average number of callers in z units of time is λz.

$$P(\Gamma_{r,\lambda} \geq z) = P(Y_{\lambda z} < r)$$

<u>Example</u>: In a radio contest, the 3rd caller every hour wins a prize. What is the probability that in a given hour, it takes at least 10 minutes to claim the prize if the average number of callers per hour is 5?

<u>Solution</u>: The average number of callers in 10 minutes is $\frac{5}{6} = \mu$.

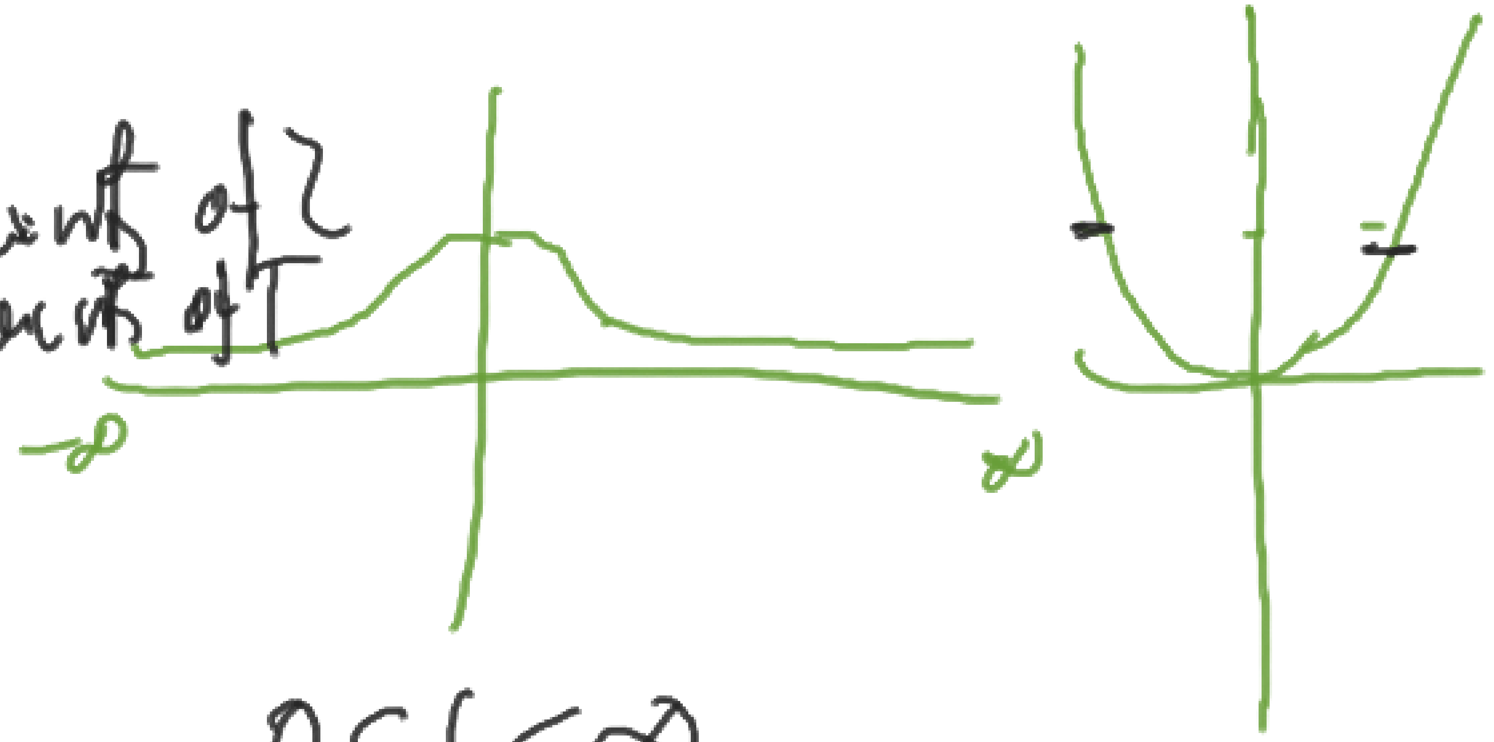
$$P(Y_\mu \leq 2) = e^{-5/6}\left(1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 \Big/ 2\right) \qquad P(Y_\mu = k) = \frac{\mu^k}{k!} e^{-\mu}$$

$= \ldots$

Example: What is the density of $T = Z^2$ where Z is the standard normal variable?

Solution: Let $f(z) = e^{-z^2/2}/\sqrt{2\pi}$ be the density of Z

$g(t)$ be the density of T



$t = z^2$ (2 branches)

$0 \le t < \infty$

$$g(t)\,dt = 2 f(z)\,dz$$

$$dt = 2z\,dz$$

$$g(t)\, 2z\, \cancel{dz} = \frac{2 e^{-z^2/2}}{\sqrt{2\pi}} \cancel{dz}$$

$$g(t) = \frac{1}{z\sqrt{2\pi}} e^{-z^2/2} \qquad z = t^{1/2}$$

$$g(t) = \frac{1}{z\sqrt{2\pi}} e^{-z^2/2} \qquad t = z^2 \qquad 0 \le t < \infty$$

$$z = \sqrt{t} = t^{1/2}$$

$$\boxed{= \begin{cases} \dfrac{t^{-1/2} e^{-t/2} \cdot (1/2)^{1/2}}{\sqrt{\pi}} & t \geq 0 \\ 0 & \text{else} \end{cases}} = \Gamma_{1/2,\,1/2}$$

$$\Gamma_{r,\lambda} : \quad \frac{t^{r-1} e^{-t\lambda} \lambda^r}{(r-1)!} \qquad \text{“} (-1/2)! = \sqrt{\pi} \text{”}$$

One defines

$$\Gamma(\alpha) = \int_0^\infty z^{\alpha-1} e^{-z}\, dz$$

$\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$, | $\Gamma(k) = (k-1)!$ if $k = 1, 2, 3, \dots$

The density of $\Gamma_{r,\lambda}$ for $r \in (0, \infty)$ is

$$\boxed{g(t) = \begin{cases} \dfrac{t^{r-1} e^{-\lambda t} \lambda^r}{\Gamma(r)} & t \geq 0 \\ 0 & \text{else} \end{cases}}$$

HW: 3 Suppose 50 callers call a center in an hour.

a) What is the precise distribution (assuming the Poisson process) of the time the 15th caller calls in?

b) Approximate this by a normal distribution.

c) What is the precise probability that the 4th caller will take at least 5 minutes to call in?