MATH 425 12/5/2022

Exam 3 in class on Friday

Review on Wednesday

Topics:
- Jointly distributed random variables, statistics, (cov, ρ)
  uncorrelated, independent random variables
- Continuous distributions: transforming the density
- basic case of convolution
- uniform distribution
- normal distribution - central limit theorem (discrete correction)
- statistical confidence - using the z-test
- (continued on next page)

- The exponential distribution (half-life etc.)
- The Gamma distribution – negative Poisson, Poisson process
- The Gamma function
- The χ^2 and t-distributions and statistical tests.

No calculators.
Supplement with the text: z-table, χ^2-table, t-table
"cheat sheet"

4 pp., you can see them on the course web page

A few words on why the χ^2-test and the t-test work and where that discussion leads to.

It requires linear algebra.

Matrix: $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$ $m \times n$ matrix $A^T = \underbrace{\begin{pmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{mn} \end{pmatrix}}_{n \times m \text{ matrix}}$

row vector: $(a_1, \ldots a_n)$
$1 \times n$ matrix

column vector $\begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}$ $m \times 1$ matrix

Matrix multiplication:

On 2×2 - matrices

$$\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 2 & 7 \end{pmatrix} = \begin{pmatrix} 2\cdot 1 + 3\cdot 2 & 2\cdot 2 + 3\cdot 7 \\ 4\cdot 1 + 5\cdot 2 & 4\cdot 2 + 5\cdot 7 \end{pmatrix} = \begin{pmatrix} 8 & 25 \\ 14 & 43 \end{pmatrix}$$

match rows $\qquad$ match columns $\qquad$ dot products of a row with a column

$$\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 2\cdot 2 + 3\cdot 5 \\ 4\cdot 2 + 5\cdot 5 \end{pmatrix} = \begin{pmatrix} 19 \\ 33 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 5 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 2\cdot 2 + 5\cdot 4 & 2\cdot 3 + 5\cdot 5 \end{pmatrix} = \begin{pmatrix} 24 & 31 \end{pmatrix}$$

unit matrix $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\qquad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(just write I)

$$IA = A = AI.$$

Determinant:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\det \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} = 2 \cdot 1 - 1 \cdot 3 = -1$$

Inverse matrix: A^{-1}

when $\det(A) \neq 0$:

Kramer rule $\quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$A^{-1}A = AA^{-1} = I$$

$$A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} \qquad \det A = 4\cdot 3 - 1\cdot 2 = 10$$

$$A^{-1} = \frac{1}{10}\begin{pmatrix} 3 & -2 \\ -1 & 4 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 3/10 & -2/10 \\ -1/10 & 4/10 \end{pmatrix}}}$$

If $X = (X_1, \dots, X_n)$ is a random vector, we have the covariance matrix

$$\operatorname{cov} X = \begin{pmatrix} \operatorname{cov}(X_1, X_1) & \operatorname{cov}(X_1, X_2) & \cdots & \operatorname{cov}(X_1, X_n) \\ \operatorname{cov}(X_2, X_1) & \operatorname{cov}(X_2, X_2) & \cdots & \operatorname{cov}(X_2, X_n) \\ \vdots & & & \\ \operatorname{cov}(X_n, X_1) & \operatorname{cov}(X_n, X_2) & \cdots & \operatorname{cov}(X_n, X_n) \end{pmatrix}$$

Expectation vector

$$E(X) = (E(X_1), E(X_2), \ldots, E(X_n)).$$

If M is a $1 \times n$ vector, A is a symmetric $n \times n$ matrix.

(positive definite: For a 1×n vector v,

$$vAv^T > 0)$$

always true for the covariance matrix

Then the normal random vector with expectation M and covariance matrix A has joint density

$$\boxed{\frac{1}{\sqrt{(2\pi)^n \det(A)}} e^{-(x-M)A^{-1}(x-M)^T/2}}$$

"A matrix form" of the density of the general normal distribution.

The goal: For a random vector, compute its expectation and covariance matrix and approximate it by a normal vector.

It gets better: There is a sense in which a normal vector is just a product of independent normal variables.

You can interpret a matrix as a linear transformation

$$\underset{\text{vector}}{v} \longmapsto \underbrace{Pv}_{\text{vector}} \quad \leftarrow \text{your matrix}$$

P is called orthogonal if $v \cdot w = Pv \cdot Pw$

dot product.

- Orthogonal transformation preserves statistics

- Every covariance matrix can be transformed into a diagonal matrix

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_n \end{pmatrix}$$

⇝ independent normal variables.

by an orthogonal transformation (replace A by $Q^{-1}AQ$ where Q is orthogonal)

How this applies to the χ^2-test.

Pearson test:

$$\sum \frac{(O_i - E_i)^2}{E_i}$$

$$X_i = \frac{O_i - E_i}{\sqrt{E_i}}$$

$p(i) = p_i$

$\sum p_i = 1$

Covariance matrix:

$$\begin{pmatrix} 1-p_1 & -\sqrt{p_1p_2} & -\sqrt{p_1p_3} & \cdots & -\sqrt{p_1p_n} \\ -\sqrt{p_2p_1} & 1-p_2 & -\sqrt{p_2p_3} & & -\sqrt{p_2p_n} \\ & & & & \\ -\sqrt{p_np_1} & -\sqrt{p_np_2} & \cdots & & 1-p_n \end{pmatrix}$$

$$= I - \underbrace{\begin{pmatrix} \sqrt{p_1} \\ \vdots \\ \sqrt{p_n} \end{pmatrix} \begin{pmatrix} \sqrt{p_1} & \cdots & \sqrt{p_n} \end{pmatrix}}_{\text{orthogonal projection on this vector}}$$

orthogonal diagonalisation

$$\downarrow$$

$$\sim \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ 0 & & & & 0 \end{pmatrix}$$

$\chi^2 =$ Σ of squares of these standard normal distributions