Note Title 8/29/2022

MATH 695 8/29/2022

<u>HW assigned each class</u>

On Fridays: type in Gradescope

due at the beginning of next class

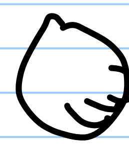
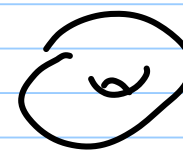
after posted (usually, next Monday 9AM,

this coming week next Wed 9AM)

Invitation code: 4VJWKD

Office hours: MWF 11AM-12PM EM 3846
(or by appointment)

Algebraic Topology

Tell spaces apart  $S^2 \not\cong$  $S^1 \times S^1$

Topology $\rightsquigarrow$ algebra

apply back to make conclusions about topology

The first step topology ⇝ algebra becomes more important: enriching algebra from topology

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Singular (co)homology of a (topological) space X.

Standard simplex

$$\Delta^n = \{(t_0, \dots t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1, \underbrace{t_i \geq 0}_{\text{this condition is aesthetic only}}\}$$

barycentric coordinates

Singular simplex in X

$$\sigma : \Delta^n \longrightarrow X$$

continuous (default assumption about maps between spaces)

$$S_n X = \{ \sigma : \Delta^n \to X \}$$

free abelian group on $S_n X$

Singular chains : $\mathbb{Z} S_n X$

$$\mathbb{Z}S = \{a : S \to \mathbb{Z} \mid \exists F \subset S \text{ finite } \forall s \in S \setminus F \quad a(s) = 0\}$$

set

$$\boxed{\underbrace{a_1 s_1 + \cdots + a_n s_n}}$$

($s_1, \ldots s_n$ different)

$$a(s_i) = a_i$$

$$a(s) = 0 \quad s \notin \{s_1, \ldots s_n\}$$

$$\prod_S \mathbb{Z} = \mathbb{Z}^S = \{a : S \to \mathbb{Z}\}$$

universal property of free ab. group

map

abelian

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, hook] & A \\ \mathbb{Z}S \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

$$s \mapsto 1 \cdot s$$

$$a_1 s_1 + \cdots a_n s_n \longmapsto a_1 f(s_1) + \cdots + a_n f(s_n)$$

More generally, if A is an abelian group

<u>n-chains with coefficients in A</u>

$$C_n(X;A) = A S_n X$$

finite

$$AS = \{ a : S \to A \mid \exists F \subset S, \ a(s) = 0 \text{ for } s \notin F \}$$

$$a_1 s_1 + \cdots + a_n s_n$$

$$a_i \in A$$

$$AS = \bigoplus_{s \in S} A$$

$$AS = \mathbb{Z}S \otimes A$$

$A \otimes B$ is the target of a universal bilinear map

ab. groups

$A \times B \longrightarrow C$

linear in each coord
(distributive)

$$\underbrace{\mathbb{Z}(A \times B)}_{a \otimes b} \Big/ \begin{array}{l} a\otimes(b+c) \\ \sim a\otimes b + a \otimes c \\ \vdots \\ \text{other rels} \end{array}$$

$$\begin{tikzcd} A \times B \arrow[r, mapsto, "\otimes"] \arrow[dr] & A \otimes B \arrow[d, dashed, "\exists !"] \\ & C \end{tikzcd}$$

n-cochains on X with coefficients in A

$$C^n(X; A) = \mathrm{Map}(S_n X, A) = A^{S_n X} = \prod_{S_n X} A$$

$$= \mathrm{Hom}(C_n X, A)$$

homomorphisms of abelian groups.

Functoriality of $C_n(X;A)$, $C^n(X;A)$ & the opposite:

let $X \xrightarrow{f} Y$ be a continuous map

$$C_n(X;A) \xrightarrow{f_*} C_n(Y;A)$$

$$a_1\sigma_1 + \cdots + a_m\sigma_m \longmapsto a_1(f\circ\sigma_1) + \cdots + a_m(f\circ\sigma_m)$$

$$\sigma_i : \Delta^n \to X$$

$$C^n(Y;A) \xrightarrow{f^*} C^n(X;A)$$

$$(\sigma : \Delta^n \to Y) \mapsto a(\sigma) \quad \longmapsto \quad (\tau : \Delta^n \to X) \mapsto a(f \circ \tau)$$

A <u>category</u> $\mathcal{C}$ class of objects $\mathrm{Obj}(\mathcal{C})$
class of morphisms $\mathrm{Mor}(\mathcal{C})$

$X \longmapsto \mathrm{Id}_X$

$X \in \mathrm{Obj}\,\mathcal{C}$ $\quad \mathrm{Id}_X \in \mathrm{Mor}\,\mathcal{C}$ $\quad \mathrm{Obj}\,\mathcal{C} \xrightarrow{\iota} \mathrm{Mor}\,\mathcal{C}$

$S, T : \mathrm{Mor}\,\mathcal{C} \to \mathrm{Obj}\,\mathcal{C}$

source and target

$$S(Id_X) = T(Id_X) = X$$

composition: $f, g \in \mathrm{Mor}\, C$

$$T(f) = S(g)$$

$$g \circ f \in \mathrm{Mor}\, C$$

$$S(g \circ f) = Sf \qquad T(g \circ f) = Tg$$

Write

$f : X \to Y$ when $S(f) = X$, $T(f) = Y$

$$\underbrace{\mathrm{Mor}_C(X, Y)}_{\boxed{\text{set}}} = \{ f \in \mathrm{Mor}\, C \mid Sf = X,\ Tf = Y \}$$

associativity, unitality of composition

sets, classes

classes which are element of another class

$$\{ X \mid X \notin X \}$$

contradiction?

Examples: Top : (topological spaces, continuous maps)

Ab : (abelian groups, homomorphisms)

$\vdots$

Note: when C is a category, I have

an *opposite category* C^{op} : reverse S, T
reverse arrows
reverse composition

$$\mathrm{Mor}_{C^{op}}(X, Y) = \mathrm{Mor}_C(Y, X)$$

A *functor* from a category C to a category D $\quad F : C \to D$
is a pair of maps $\quad F = \mathrm{Obj}(F) : \mathrm{Obj}\, C \to \mathrm{Obj}\, D$
$\quad F = \mathrm{Mor}(F) : \mathrm{Mor}\, C \to \mathrm{Mor}\, D$

preserving r, S, T, composition.

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variable not written (Bourbaki)

<u>Examples</u> : $\mathbb{Z}?$: Sets $\to$ Ab

free ab. group

$A?$: Set $\to$ Ab $\quad$ (A ab. group)

$C_n(?;A)$: Top $\to$ Ab

A <u>contravariant functor</u> from C to D is a functor $F : C^{op} \to D$ (equivalently $F : C \to D^{op}$)

<u>Example</u> : $C^n(?; A) : \mathrm{Top}^{op} \to \mathrm{Ab}$

$\mathrm{Map}(?; A) : \mathrm{Set}^{op} \to \mathrm{Ab}$

HW :

① Prove that

$$A \otimes (B \oplus C) \cong (A \otimes B) \oplus (A \otimes C)$$

for abelian groups A, B, C.

(Recall $B \oplus C = B \times C$.)