Note Title 8/31/2022

MATH 695 8/31/2022

$\mathcal{C}$ Category : $\mathrm{Obj}\,\mathcal{C}$, $\mathrm{Mor}\,\mathcal{C}$

$F : \mathcal{C} \to \mathcal{D}$ <u>Functor</u> $F : \mathrm{Obj}\,\mathcal{C} \to \mathrm{Obj}\,\mathcal{D}$

$F : \mathrm{Mor}\,\mathcal{C} \to \mathrm{Mor}\,\mathcal{D}$

$F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$ $S(F(f)) = F(S(f))$

$T(F(f)) = F(T(f))$

$F(f \circ g) = F(f) \circ F(g)$

If $F, G : C \to D$, what is a "morphism from F to G"? <u>Natural transformation</u>:

$\varphi : F \to G$

For every object $X \in \mathrm{Obj}\, C$, we have a morphism

$$\varphi_X : F(X) \to G(X) \in \mathrm{Mor}\, D$$

For a morphism $f : X \to Y \in \mathrm{Mor}(C)$

$$\begin{tikzcd}
F(X) \arrow[r, "\varphi_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[r, "\varphi_Y"] & G(Y)
\end{tikzcd}$$

commutes:

$G(f) \circ \varphi_X$

$= \varphi_Y \circ F(f)$.

(In HW from last time, most likely you will produce natural transformations both ways and prove that they are inverse.

However, it is not required. Hint to produce them: use universal property of quotient.)

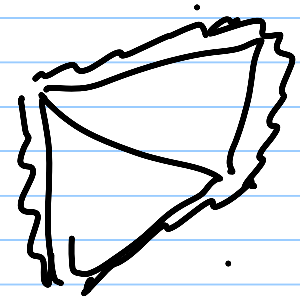
Back to singular (co)homology

$$\mathbb{Z}\{\sigma : \Delta^n \longrightarrow X\} = C_n X$$

$$C_n(X; A) = C_n X \otimes A \quad : \mathrm{Top} \to \mathrm{Ab}$$

$$C^n(X; A) = \mathrm{Hom}(C_n X, A) : \mathrm{Top}^{\mathrm{op}} \to \mathrm{Ab}$$

"Boundary of a chain"



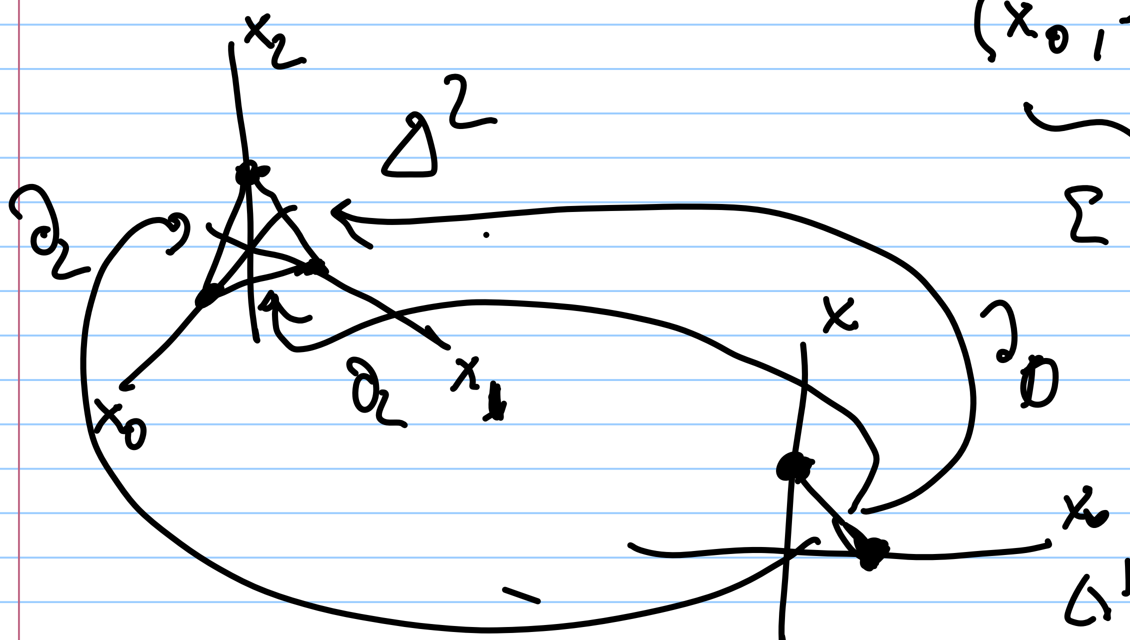
c: Σ of these singular simplices

dc = "boundary of c"

Note : <u>orientation</u>

We want a definition where $d \circ d = 0$.

Standard mappings: $\partial_i : \Delta^{n-1} \longrightarrow \Delta^n \quad i = 0, \dots, n$



$$\underbrace{(x_0, \dots, x_{n-1})}_{\sum x_j = 1} \longmapsto \underbrace{(x_0, \dots x_{i-1}, 0, x_i, \dots, x_{n-1})}_{\sum = 1}$$

$$0 \leq i \leq j \leq n$$

$$\partial_i \circ \partial_j = \partial_{j+1} \circ \partial_i$$

$$d_n = d : C_n X \longrightarrow C_{n-1} X$$

$$\sigma : \Delta^n \to X \longmapsto \sum_{i=0}^{n} (-1)^i (\sigma \circ \partial_i)$$

$$\Delta^{n-1} \xrightarrow{\partial_i} \Delta^n \xrightarrow{\sigma} X$$

Lemma: $dd = 0$

Proof: $\sigma : \Delta^n \to X$

$$dd(\sigma) = d\left(\sum_{i=0}^{n} (-1)^i \sigma \circ \partial_i\right) =$$

$$= \underbrace{\sum_{i=0}^{n} \sum_{j=0}^{n-1}}_{(i,j)} (-1)^{i+j} \sigma \circ \underbrace{\partial_i \circ \partial_j}_{} = 0. \qquad \square$$

opposite signs

bijective $(i,j) \updownarrow (j+1, i)$ on $0 \leq i \leq j \leq n-1 \updownarrow 0 \leq j < i \leq n$

Because $? \otimes A$ is functor,

$$d = d_n = d_n \otimes A \; : \; C_n(X;A) \longrightarrow C_{n-1}(X;A)$$

$$d_{n-1} \circ d_n = dd = 0$$

$$d = d^n = \mathrm{Hom}(d_{n+1}, A) \; : \; C^n(X;A) \longrightarrow C^{n+1}(X;A)$$

$$d^{n+1} \circ d^n = 0$$

Definition: **Chain complex** (of abelian groups):
A system of abelian groups $C = (C_n,\ n \in \mathbb{Z})$,
together with homomorphisms $d = d_n : C_n \to C_{n-1}$
such that $d \circ d = 0$.

Cochain complex $C^* = (C^n,\ n \in \mathbb{Z})$
together with homomorphisms $d = d^n : C^n \to C^{n+1}$
such that $d \circ d = 0$.

From a chain complex, I can always make a cochain complex

$$C^n = C_{-n}$$

and vice versa. This is an example of an <u>equivalence of categories</u>:

Two categories $\mathcal{C}, \mathcal{D}$ are equivalent when there are functors $F: \mathcal{C} \to \mathcal{D}$,

$G : \mathcal{D} \to \mathcal{C}$ and $\underbrace{\text{natural isomorphisms}}_{\text{natural transformation with an inverse}}$

$$G \circ F \cong \mathrm{Id}_{\mathcal{C}}$$

$$F \circ G \cong \mathrm{Id}_{\mathcal{D}}.$$

We define homology of a chain complex C

$$H_n C := \operatorname{Ker}(d_n : C_n \to C_{n-1}) / \operatorname{Im}(d_{n+1} : C_{n+1} \to C_n)$$

$$d_n^{-1}(0) \supseteq d_{n+1}(C_{n+1})$$

$d_n \circ d_{n+1} = 0$

For a cochain complex C^*

$$H^n C^* := \operatorname{Ker}\left(d^n : C^n \to C^{n+1}\right) \Big/ \operatorname{Im}\left(d^{n-1} : C^{n-1} \to C^n\right)$$

<u>Singular homology and cohomology</u> of a space X with coefficients in an abelian group A:

$$H_n(X;A) := H_n(C(X;A)) = H_n(C(X)\otimes A)$$

$$C(X) = (C_n X,\ d_n)$$

$$\uparrow$$

$= 0$ when $n < 0$

$$H^n(X;A) := H^n(\mathrm{Hom}(C(X),A)) = H^n(C^*(X;A))$$

$$C^*(X;A) = (\mathrm{Hom}(C_n(X),A),\ d^n)$$

$$\mathrm{Hom}(d_{n+1}, A).$$

$H_n(?;A) : \mathrm{Top} \to \mathrm{Ab}$

$H^n(?;A) : \mathrm{Top}^{op} \to \mathrm{Ab}$

functors.

HW: (1) Prove that the following two categories are equivalent:

$$\begin{tikzcd} X \arrow[loop left, "Id_X"] \end{tikzcd}$$

$$\begin{tikzcd} Y \arrow[loop left, "Id_Y"] \arrow[r, bend left, "f"] & Z \arrow[l, bend left, "g"] \arrow[loop right, "Id_Z"] \end{tikzcd}$$

$$g \circ f = Id_Y$$
$$f \circ g = Id_Z.$$

(2) A group G can be made into a category in the following way: There is only one object X, and the

element of G are morphisms $g : X \to X$.
Prove that when G is a non-trivial group, this category is not equivalent to

$$\begin{tikzcd} X \arrow[loop right, "\mathrm{Id}"] \end{tikzcd}$$

$G \neq \{e\} \Rightarrow \mathcal{C}_G$ is not equivalent to $\mathcal{C}_{\{e\}}$