MATH 695 9/2/2022

Note Title 9/2/2022

CX : $\cdots \xrightarrow{d} C_n X \xrightarrow{d} C_{n-1} X \xrightarrow{d} \cdots$ chain cx.

X space $d\sigma = \sum_{i=0}^{n} (-1)^i (\sigma \circ \partial_i)$

$C_n X = \mathbb{Z}\{\sigma : \Delta^n \to X\}$ $\partial_i : \Delta^{n-1} \to \Delta^n$.

$C(X;A)$: $\xrightarrow[d_{n+1}]{d\otimes A} \underbrace{C_n X \otimes A}_{C_n(X;A) \cong A\{\sigma : \Delta^n \to A\}} \xrightarrow[d_n]{d\otimes A} C_{n-1} X \otimes A$

A ab. group

$C(X)$, $C(X;A)$ chain cxes

$$H_n(X;A) = H_n C(X;A) = \operatorname{Ker} d_n / \operatorname{Im} d_{n+1}$$

↑ singular homology

$$C^*(X;A) : \cdots \leftarrow \operatorname{Hom}(C_{n+1}X; A) \xleftarrow{d^n} \underbrace{\operatorname{Hom}(C_nX; A)}_{\operatorname{Map}(\{\sigma : \Delta^n \to X\}, A)} \leftarrow \cdots$$

$$= A^{\{\sigma : \Delta^n \to X\}}$$

$$c : \{\sigma : \Delta^n \to X\} \longrightarrow A$$

$$(d^n c)(\tau : \Delta^{n+1} \to X) = c(d_n \tau) =$$

$$= c\left(\sum_{i=0}^{n+1}(-1)^i \tau\circ\partial_i\right) = \sum_{i=0}^{n+1}(-1)^i c(\tau\circ\partial_i)$$

$\mathbb{Z}$-Chain $=$ category of $\underbrace{\text{chain complexes}}_{\text{objects}}$

morphisms $=$ chain maps

$$C = \left(\to \cdots \to C_n \xrightarrow{d} C_{n-1} \to \cdots\right)$$

$$D = \left(\to \cdots \to D_n \xrightarrow{d} D_{n-1} \to \cdots\right)$$

A <u>chain map</u> $f : C \to D$: For every $n \in \mathbb{Z}$

$f_n (= f) : C_n \to D_n$

$$\begin{tikzcd} C_n \arrow[r, "f_n"] \arrow[d, "d"'] & D_n \arrow[d, "d"] \\ C_{n-1} \arrow[r, "f_{n-1}"] & D_{n-1} \end{tikzcd}$$

commutes.

If R is, say, a commutative ring, we have analogously a category R-Chain of chain complexes of R-modules and chain maps of R-modules.

Similarly, we have an equivalent category to R-chain R-Cochain where the differentials go up.

H_n : $\mathbb{Z}$-chain $\longrightarrow$ Ab

R-chain $\longrightarrow$ R-Mod $\leftarrow$ R-Modules $\leftarrow$ commutative ring

$H^n : \mathbb{Z}\text{-Cochain} \to \mathrm{Ab}$ <u>covariant</u>

(similarly R) (not contravariant)

$$\begin{tikzcd}
\mathrm{Top}^{op} \arrow[r, "C^*(-;A)"] \arrow[rr, bend right] & \mathbb{Z}\text{-Cochain} \arrow[r, "H^n"] & \mathrm{Ab}
\end{tikzcd}$$

(Co) singular homology is <u>computable</u>

Eilenberg - Steenrod axioms

(Co) Chain complex of <u>spaces</u>

Pair

category

Objects = pairs (X, Y) of spaces where $Y \subseteq X$

subspace topology

Morphisms $(X,Y) \to (X',Y')$: $f: X \to X'$ continuous, $f(Y) \subseteq Y'$.

(Think of X/Y.

quotient topology

This is not always well behaved.

$\mathbb{R}/\mathbb{Q}$.)

quotient topological space

Define for a pair (X,Y) $\quad C(X,Y) := \underbrace{C(X)/C(Y)}$ chain complexes

$D \subseteq C$

chain map which is an inclusion $D_n \subseteq C_n$

induced d_n

Define C/D : $\to C_n/D_n \to C_{n-1}/D_{n-1}$

HW: (4) Suppose $S \subseteq T$ is an inclusion of sets.

Prove that $\mathbb{Z}(T \setminus S) \cong \mathbb{Z}T / \mathbb{Z}S$.

For a pair (X, Y), $C(X, Y)$ is a chain complex of free abelian groups. (Meaning each $C_n(X, Y)$ is a free abelian group.)

$$C(X, Y; A) := C(X, Y) \otimes A = (\to \to C_n(X, Y) \otimes A \to \cdots)$$

$$C^*(X, Y; A) := \mathrm{Hom}(C(X, Y), A) = (\leftarrow \mathrm{Hom}(C_n(X, Y), A) \leftarrow)$$

$H_n(X,Y;A) := H_n(C(X,Y;A))$ $\quad$ $\mathrm{Pair} \to \mathrm{Ab}$

$H^n(X,Y;A) := H^n(C^*(X,Y;A))$ $\quad$ $\mathrm{Pair}^{op} \to \mathrm{Ab}.$

<u>Exact sequence</u>

$$\to \cdots A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \to \cdots \qquad \operatorname{Ker} d_n = \operatorname{Im} d_{n+1}$$

(like a chain complex with homology 0, not necessarily indexed over $\mathbb{Z}$. When it ends, where it ends, no condition).

Short exact sequence $\quad 0 \to A \xrightarrow{\subseteq} B \xrightarrow{\text{onto}} C \to 0$

"$C = B/A$."

A key point of homological algebra:

If $\quad 0 \to C' \xrightarrow{i} C \xrightarrow{j} C'' \to 0$

is a short exact sequence of chain complexes

(i.e. $\quad C'' = C/C'$)

Then we get a long exact sequence in homology:

$$\cdots \to H_n C' \xrightarrow{i_* = H_n(i)} H_n C \xrightarrow{j_* = H_n(j)} H_n C'' \xrightarrow{\partial} H_{n-1} C' \xrightarrow{i_* = H_{n-1}(i)} H_{n-1} C \to \cdots$$

connecting map,

natural in pairs of chain complexes $C' \subseteq C$.

Main point (constructing ∂):

$$
\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & C'_n \arrow[r] \arrow[d, "d"'] & C_n \arrow[r] \arrow[r, dashed, bend right, mapsto] \arrow[d, "d"'] \arrow[d, dashed, bend left] & C''_n \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & C'_{n-1} \arrow[r] \arrow[r, dashed, bend right, mapsto] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & C''_{n-1} \arrow[r] \arrow[d] & 0 \\
 & {} & {} & {} &
\end{tikzcd}
$$

.....

$\square$

Terminology:

cycle $\in$ Ker d

Boundary $\in$ Im d

Eilenberg - Steenrod axioms

① Functoriality : $H_n(?;A)$: $\mathrm{Top} \to \mathrm{Ab}$

$\mathrm{Pair} \to \mathrm{Ab}$

$H^n(?;A)$: $\mathrm{Top}^{op} \to \mathrm{Ab}$

$\mathrm{Pair}^{op} \to \mathrm{Ab}$

② Homotopy : $f, g : X \to X'$

$f \simeq g = h : X \times [0,1] \to X'$ | for pairs $(X,Y) \to (X',Y')$

homotopy $\quad h(x,0) = f(x)$ | $h(y,t) \in Y$
$h(x,1) = g(x)$ | $y \in Y$

Homotopic maps induce the same map in homology and cohomology.

$f \simeq g$: $\underset{H_n f}{\overset{\|}{f_*}} = \underset{H_n g}{\overset{\|}{g_*}} : H_n(X;A) \to H_n(Y;A)$

similarly for pairs $\quad \underset{H^n f}{\overset{\|}{f^*}} = \underset{H^n g}{\overset{\|}{g^*}} : H^n(Y;A) \to H^n(X;A).$

③ Exactness axiom: For a pair (X,Y) $Y \subseteq X$

we have a natural long exact sequence

$$\to H_n(Y;A) \xrightarrow{\subseteq_*} H_n(X;A) \xrightarrow{\subseteq_*} H_n(X,Y;A) \xrightarrow{\partial} H_{n-1}(Y;A) \to \cdots$$

$(X,\emptyset)$

$$\to H^n(X,Y;A) \xrightarrow{\subseteq^*} H^n(X;A) \to H^n(Y;A) \xrightarrow{\delta} H^{n+1}(X,Y;A) \to$$

④ Excision axiom: $Z \subset Y \subset X$

$\text{Closure}_X(Z) \subseteq \text{Interior}_X(Y)$

Then the inclusion of pairs

$$(X \setminus Z, Y \setminus Z) \subseteq (X, Y)$$

induces $\cong$ in (co)homology

$$H_n(X \setminus Z, Y \setminus Z; A) \xrightarrow{\cong} H_n(X, Y; A)$$

$$H^n(X, Y; A) \xrightarrow{\cong} H^n(X \setminus Z, Y \setminus Z; A)$$

(5) $H_n(*;A) \cong H^n_1(*,A) = \begin{cases} A & n=0 \\ 0 & \text{else} \end{cases}$

$*$ = single point.

HW (ungraded): Read the proofs of the homotopy and excision axiom on the course webpage.