MATH 695 9/7/2022

Note Title 9/7/2022

Limit axioms: X_i, $i \in I$ top. spaces, A ab. group

$$\bigoplus_{i \in I} H_n(X_i; A) \xrightarrow[\bigoplus_{i\in I} \subseteq_*]{\cong} H_n\left(\coprod_{i \in I} X_i; A\right)$$

$$H^n\left(\coprod_{i \in I} X_i; A\right) \xrightarrow[\prod_{i\in I} \subseteq^*]{\cong} \prod_{i \in I} H^n(X_i, A)$$

$X_1 \subseteq X_2 \subseteq X_3 \subseteq \dots$ Hausdorff spaces

$X = \bigcup X_n$ (union topology) induced topology

$Z \subseteq X$ closed iff $Z \cap X_i$ closed in X_i

$$\begin{tikzcd}
H_n(X_1;A) \arrow[r] \arrow[drr] & H_n(X_2;A) \arrow[r] \arrow[dr] & H_n(X_3;A) \arrow[r] \arrow[d] & \dots \\
 & & H_n(X;A) &
\end{tikzcd}$$

$$\operatorname{colim} H_n(X_i;A) \xrightarrow{\cong} H_n(X;A)$$

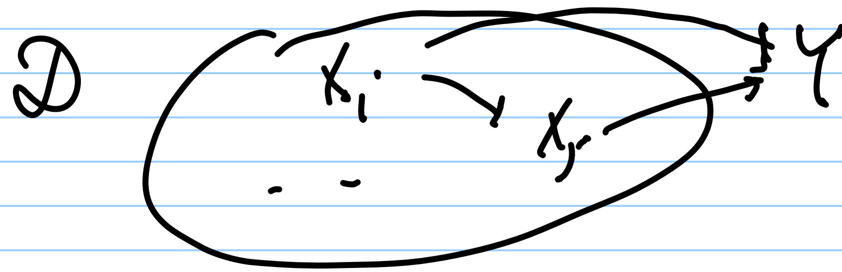
In a category C, a <u>diagram</u> is a functor $D : I \to C$ where I is a small category (objects form a set ∴ morphisms form a set)

<u>Example</u>:

$$I : \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \\ Z & \end{tikzcd}$$

a diagram D

A colimit of a diagram $\mathcal{D}$ (if one exists)



is an object Y together with compatible [commuting with arrows inside diagram] morphisms $X_i \to Y$ such that Y is universal among such data:

$\exists!$ morphism which makes everything commute.

<u>Example:</u> $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \longrightarrow \cdots$

a sequence of abelian groups.

$$\underset{\overrightarrow{f_i}}{\operatorname{colim}} A_i = \bigoplus_{i \in \mathbb{N}} A_i \Big/ (a \in A_i) \sim (f_i(a) \in A_{i+1})$$

<u>Example</u>: A. diagram w/o arrows (except Identities)

colim A_i = $\coprod A_i$

$\uparrow$ coproduct

examples: sets − disjoint union

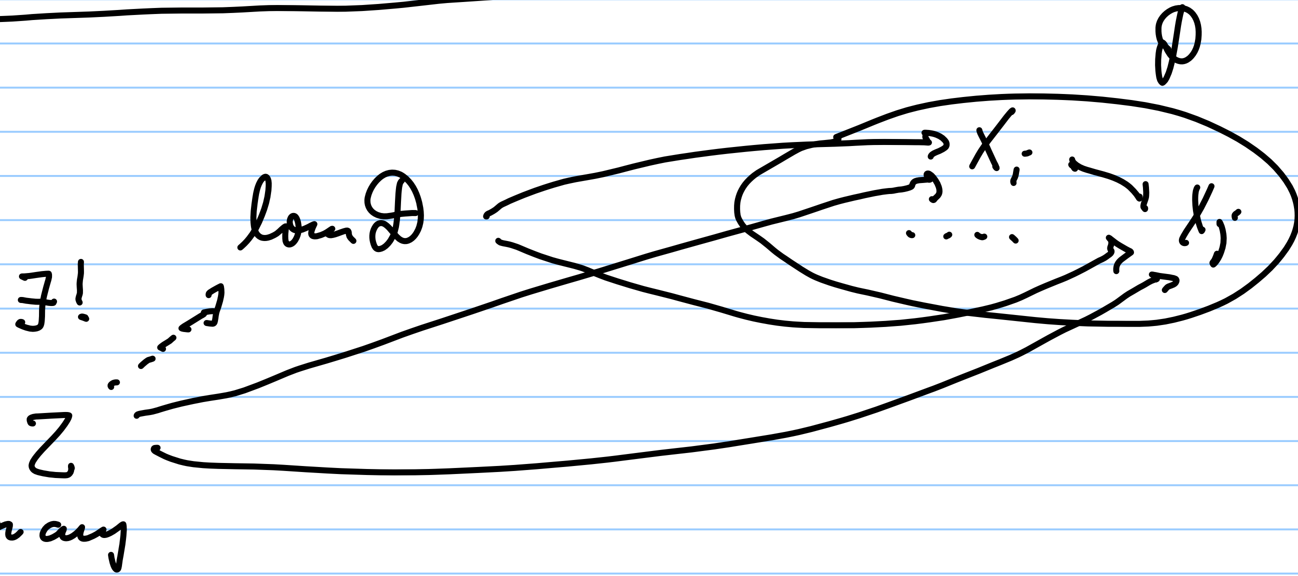
groups − free product $*$

ab. groups − $\oplus$

empty diagram: initial object i

$\exists !$ morphism from Z to another object

Dual notion: limits



Example: $\cdots \to A_4 \xrightarrow{f_3} A_3 \xrightarrow{f_2} A_2 \xrightarrow{f_1} A_1$

$$\varprojlim_{f_i} A_i = \{(a_i) \in \prod_i A_i \mid f_i(a_{i+1}) = a_i\}$$ ab. groups

limit of a diagram w/o arrows: **product**

limit of empty diagram = terminal object t

For every object X $\exists!$ $X \to t \in \mathrm{Mor}(\mathcal{C})$

A property of the colimit of a sequence of abelian groups is that it preserves exactness:

$$\begin{array}{ccccc} 0 & & 0 & & \\ \downarrow & & \downarrow & & \\ A_1 & \to & A_2 & \to & \cdots \\ \downarrow & & \downarrow & & \\ B_1 & \to & B_2 & \to & \cdots \\ \downarrow & & \downarrow & & \\ C_1 & \to & C_2 & \to & \cdots \\ \downarrow & & \downarrow & & \\ 0 & & 0 & & \end{array} \quad \rightsquigarrow \quad \begin{array}{c} 0 \\ \downarrow \\ \mathrm{colim}\, A_i \\ \downarrow \\ \mathrm{colim}\, B_i \\ \downarrow \\ \mathrm{colim}\, C_i \\ \downarrow \\ 0 \end{array}$$

A similar statement is not true for limits

$$\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] \\
\lim A_i \arrow[d] & \cdots \arrow[r] & A_2 \arrow[r] \arrow[d] & A_1 \arrow[d] \\
\lim B_i \arrow[d] & \cdots \arrow[r] & B_2 \arrow[r] \arrow[d] & B_1 \arrow[d] \\
\lim C_i & \cdots \arrow[r] & C_2 \arrow[r] \arrow[d] & C_1 \arrow[d] \\
 & & 0 & 0
\end{tikzcd}$$

not necessarily onto.

HW ①

$$
\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
\cdots \arrow[r] & \mathbb{Z} \arrow[r, "p"] \arrow[d, "p^k"] & \cdots & \mathbb{Z} \arrow[r, "p"] \arrow[d, "p^2"] & \mathbb{Z} \arrow[r, "p"] \arrow[d, "p"] & \mathbb{Z} \arrow[d, "1"] \\
\cdots \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d] & \cdots & \mathbb{Z} \arrow[r, "1"] \arrow[d] & \mathbb{Z} \arrow[r, "1"] \arrow[d] & \mathbb{Z} \arrow[d] \\
\cdots \arrow[r] & \mathbb{Z}/p^k \arrow[r, "\text{onto}"] \arrow[d] & \cdots & \mathbb{Z}/p^2 \arrow[r] \arrow[d] & \mathbb{Z}/p \arrow[r] \arrow[d] & 0 \arrow[d] \\
 & 0 & & 0 & 0 & 0
\end{tikzcd}
$$

Show that the limit is not exact.

This has to do with the fact that

$$X_1 \subseteq X_2 \subseteq \cdots \qquad X = \bigcup X_i$$

$$H^n(X;A) \to H^n(X_i;A)$$

But we may not necessarily have

$$H^n(X;A) \xrightarrow{\cong} \lim H^n(X_i;A)$$

Recall the dimension axiom: $H_i(*;A) =$
$H^i(*;A) = A \quad i=0$
0 else

If we drop this axiom, we get the notion of a generalised homology and cohomology theory (a collection of functors E_n, resp. E^n
which satisfy all the axioms as $H_n(?;A)$,

$H^*(\ \cdot\ ;A) \leftarrow$ ordinary homology (coh.)

except the dimension axiom.

Next: Computing ordinary (co)homology

- CW complexes

→ spectral sequence

Based spaces = pairs of the form $(X,*)$

(Pointed)

(a space with a chosen point)

A category has a 0 if it has an initial object i and a terminal object t and the unique morphism

$$i \longrightarrow t$$

is an isomorphism

If E is a generalized (co)homology theory and $X = (X, *)$ is a based space, then reduced homology and cohomology

$$\begin{cases} \tilde{E}_n(X) := E_n(X, *) \\ \tilde{E}^n(X) := E^n(X, *) \end{cases}$$

in particular, for $E = H(\,?\,; A)$.

HW (2) Prove that for a based space X, the inclusion of a base point induces a splitting

$$E_n(X) = \tilde{E}_n(X) \oplus E_n(*)$$

$$E^n(X) = \tilde{E}^n(X) \oplus E^n(*)$$