MATH 695 9/9/2022

Note Title 9/9/2022

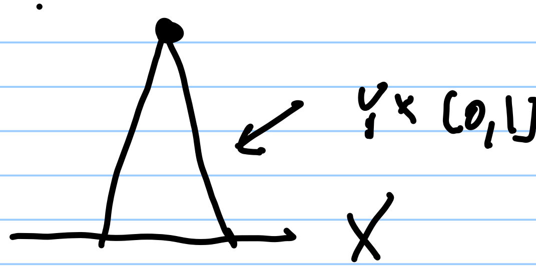
Caution: For a general pair (X,A), we do not have $E_n(X,A) \cong \tilde{E}_n(\underbrace{X/A}_{\substack{\text{replace this}\\\text{by a geometric}\\\text{construction where}\\\text{it does work.}}})$.

Given a map $f : Y \to X$, the <u>mapping cone</u>

(continuous)

is defined as

$$Cf := \left(X \amalg (Y \times [0,1])\right) \Big/ \begin{matrix} (y,1) \sim (y',1) \\ (y,0) \sim f(y) \end{matrix}$$



$$CY := C(Id : Y \to Y) = Y \times [0,1] \,/\, (y,1) \sim (y',1)$$

$$Cf = \operatorname{colim}\left(\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "0"'] & X \\ Y \times [0,1] & \\ Y \arrow[u, "1"] \arrow[r] & * \end{tikzcd} \right)$$

$$y \mapsto (y,0), \qquad y \mapsto (y,1)$$

$$\begin{tikzcd} * \arrow[r, hook] \arrow[d, phantom, "\shortparallel"] & Cf \\ (y,1) & \end{tikzcd}$$

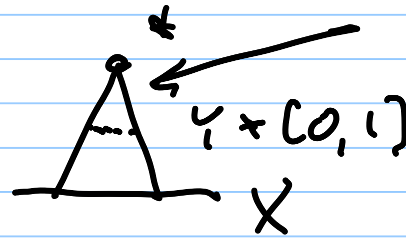
Proposition: If E is a generalised (co)homology and (X,Y) is a pair, $i : Y \hookrightarrow X$ then we have a natural isomorphism

$$E_n(X,Y) \cong \tilde{E}_n(C_i)$$

resp.

$$E^n(X,Y) \cong \tilde{E}^n(C_i)$$

Proof: (for gen. homology) C_i



$D \subset C_i$

$D = \{(y,t) \mid t \geq \frac{1}{2}\}$

$\cancel{(y,1) \sim (y',1)}$

$$\tilde{E}_n(Cf) \overset{def}{=} E_n(Cf, *)$$

$\cong$ homotopy axiom

$$E_n(Cf, D)$$

excision $\cong$

$$E_n(Cf \setminus \{*\}, D \setminus \{*\})$$

$$\cong$$

$$E_n(X, Y)$$

$D \simeq *$ ← contractible

homotopy equivalent

($X \underset{g}{\overset{f}{\rightleftarrows}} Y$ is a homotopy equivalence if $fg \simeq Id_Y$
$gf \simeq Id_X$

hTop = homotopy category
objects = top. spaces
morphisms = homotopy

$D \setminus \{*\} \simeq Y$

$Cf \setminus \{*\} \simeq X$

X, Y homotopy equivalent means: $\cong$ in hTop

{classes of maps} equivalence relation of being homotopic

$\square$

Comments: Often we can avoid discussing homotopy equivalence of pairs by the 5-lemma:

If I have a diagram where the rows are exact

$$
\begin{tikzcd}
\cdots \arrow[r] & \bullet \arrow[r] \arrow[d, "\cong"'] & \bullet \arrow[r] \arrow[d, "\cong"'] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d, "\cong"] & \bullet \arrow[r] \arrow[d, "\cong"] & \cdots \\
\cdots \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \cdots
\end{tikzcd}
$$

If these are isomorphisms

then the one in the middle is an isomorphism.

(typically in a LES for homology:

A map of pairs $(X, Y) \to (X', Y')$ where $X \to X'$, $Y \to Y'$ are homotopy equivalences the map of pairs induces an $\cong$

in (co)homology (apply 5-lemma to LES).
in homology

We can ask: When can we say that $E_n(X,Y) \cong \tilde{E}_n(X/Y)$ or similarly for cohomology?

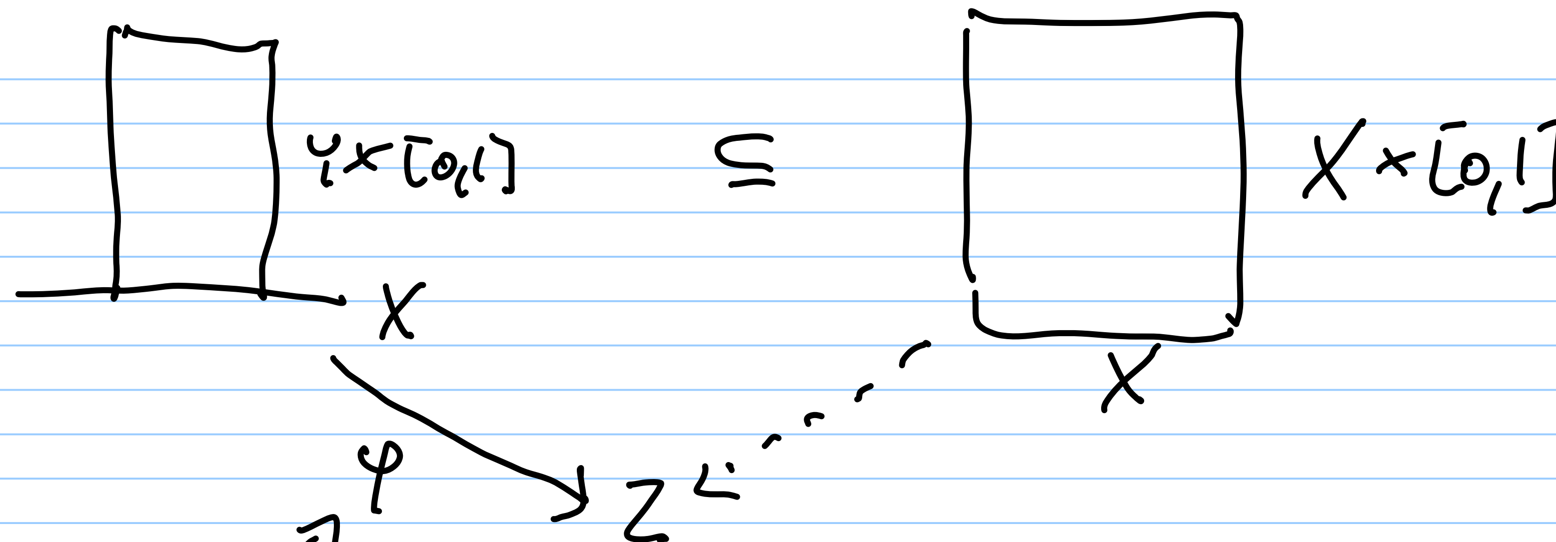
$i : Y \to X$ inclusion

Answer: When $Ci \xrightarrow{\simeq} X/Y$ is a homotopy equivalence.

$(y,t) \longmapsto *$

A Cofibration $i: Y \to X$ is an inclusion which satisfies the homotopy extension property:

If I have any map $f: X \to Z$ and a homotopy $h: Y \times [0,1] \to Z$ $h(y,0) = f(y)$ then it extends to a homotopy $H: X \times [0,1] \to Z$ with $H(x,0) = f$.



M_f

$\uparrow$

mapping cylinder : $X \sqcup Y\times[0,1] \,/\, (y,0)\sim f(y)$

It suffices to consider $\varphi = \mathrm{Id}_{M_f}$

The HEP (cofibration condition) is equivalent to $M_i \subseteq X \times [0,1]$ being a retract = (having a left inverse)

(Example - will talk about soon - CW pair cell pair)

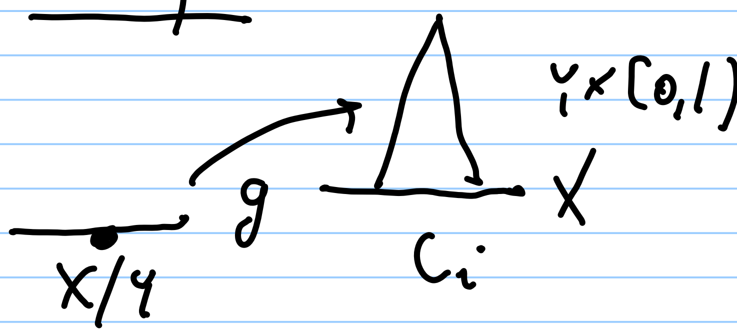
<u>Proposition</u>: If $i: Y \to X$ is a cofibration

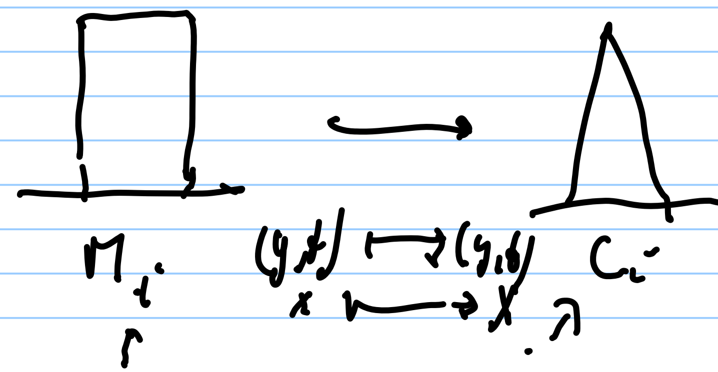
then $\gamma : C_i \to X/Y$ is a homotopy equivalence.

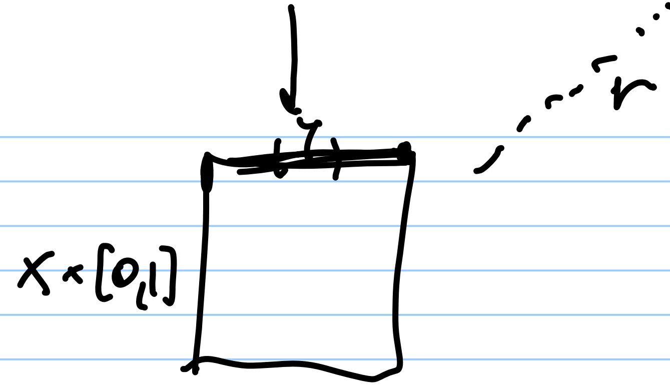
$$\begin{aligned} x &\longmapsto x \\ (y,t) &\longmapsto * \end{aligned}$$

$$\boxed{\begin{array}{l}\text{Then} \\ E_n(X,Y) \cong \tilde{E}_n(X/Y) \\ E^n(X,Y) \cong \tilde{E}^n(X/Y)\end{array}}$$

Proof:





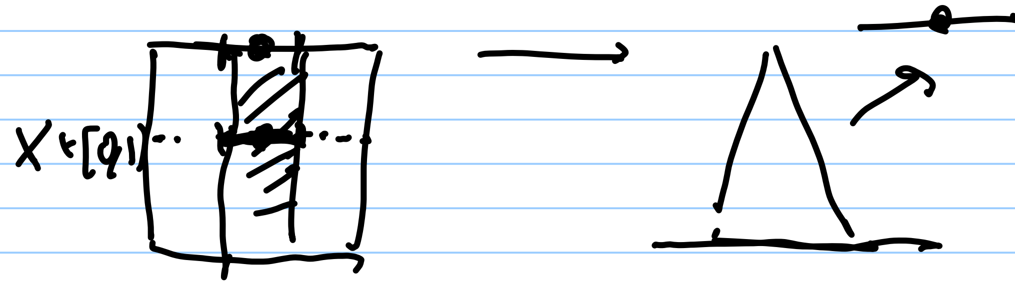


$g(x) = r(x,1)$

$g \circ \gamma \simeq Id_{C_i}$

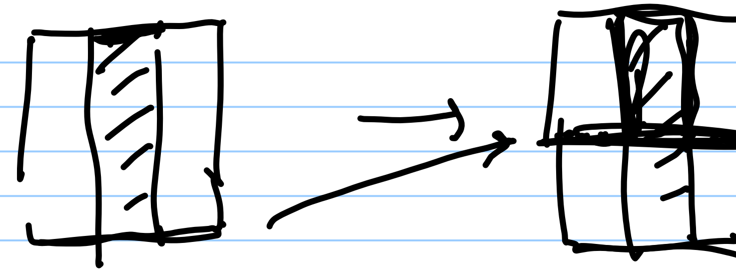
$\gamma \circ g \simeq Id_{X/\gamma}$

$(x,t) \mapsto \gamma r(x,t)$



shorten the cylinder
at time t
linear homothety
$[0,1] \longrightarrow [t,1]$

identify $(y,1) \simeq *$.



$\square$

HW: (3) The unreduced suspension $SX = C(X \to *)$.
(a) Describe the unreduced suspension directly.

(b) Prove that $\tilde{E}_{n+1} SX \cong \tilde{E}_n X$ when X has a base point.

<u>Note</u>: You are allowed to use long exact sequence in reduced homology: If $\begin{array}{ccc} Y & \overset{i}{\hookrightarrow} & X \\ * & \longmapsto & * \end{array}$ is a based inclusion, then we have a LES

$$\to \tilde{E}_n Y \xrightarrow{i_*} \tilde{E}_n X \to E_n(X, Y) \to \tilde{E}_{n-1}(Y) \to \cdots$$

(do not have to prove this)