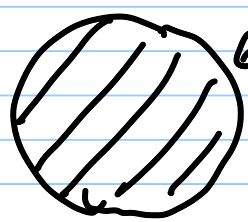
MATH 695 9/12/2022

Note Title 9/12/2022

CW-complex, cell complex

The idea to capture: a space made by gluing cells to existing stages of the construction

A cell $= (D^n, S^{n-1})$.



glued by the boundary.

$D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$

$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$

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CW-complex means the cells are only glued to cells of lower dimension (not required in a cell complex)

(every cell complex is homotopy equivalent to a CW-complex)

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in algebraic topology

Key point: we are mostly interested in CW-complexes. For any topological space, we will introduce an "approximation" by a CW-complex (not necessarily a homotopy equivalence) which will preserve all the invariants we introduce.

Definition: A **CW-complex** X is defined as follows:

subspace topologies

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \qquad X = \bigcup X_k$$

$\leftarrow$ skeleta

union topology

We have a set of k-cells I_k and an **attaching map**
$\varphi_k : I_k \times S^{k-1} \to X_{k-1}$, X_k is the colimit (pushout)

discrete

of the diagram

$$\left(\begin{tikzcd} I_k \times S^{k-1} \arrow[r, "\varphi_k \times S^{k-1}"] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & X_{k-1} \arrow[d, dotted] \\ I_k \times D^k \arrow[r, dotted] & X_k \end{tikzcd}\right)$$

$$X_k = \left(I_k \times D^k\right) \amalg X_{k-1} \Big/ \varphi_k(s) \sim s \quad s \in I_k \times S^{k-1}$$

A cell complex X is defined more generally

$$\emptyset = X_{(-1)} \subseteq X_{(0)} \subseteq X_{(1)} \subseteq \cdots \qquad X = \bigcup X_{(k)}$$

J_k = indexing set $\quad d_k : J_k \longrightarrow \mathbb{N}_0$

dimension function

$D^0 = *$

$S^{-1} = \emptyset$

$X_{(k)} =$ Pushout : $\varphi_k : \coprod_{j \in J_k} S^{d_k(j)-1} \to X_{(k-1)}$

$$\begin{tikzcd} \coprod_{j \in J_k} S^{d_k(j)-1} \arrow[r, "\varphi_k"] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & X_{(k-1)} \arrow[d] \\ \coprod_{j \in J_k} D^{d_k(j)} \arrow[r] & X_{(k)} . \end{tikzcd}$$

A <u>CW-pair</u> (X, Z) is defined the same way as a CW-complex with the exception that $X_{-1} = Z$.
(cell pair analogously)

<u>Proposition</u>: A cell pair is a <u>cofibration</u> (satisfies the homotopy extension property).

<u>Proof</u>:

HW 1 $\quad S^{k-1} \overset{i}{\hookrightarrow} D^k$ is a cofibration.

(Hint: Draw the inclusion $M_i \subset D^k \times [0,1]$.
This is homeomorphic to another pair which is
obviously a retraction.)

The homotopy extension property is preserved
under colimits: disjoint unions, increasing unions
and pushouts. For example for pushouts, if

$$Z \overset{i}{\subseteq} Y \quad \text{is a cofibration}$$

$f : Z \to X$ any (cont.) map, then

Lemma: If $i: A \to X$ is a cofibration and $g: A \to B$ is any map then the pushout $B \to B \amalg_A X$ is a cofibration.

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d] \\ B \arrow[r] & B \amalg_A X \end{tikzcd}$$

(May, p. 42).

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\begin{tikzcd}
A \arrow[rrrr] \arrow[dr, "g"] \arrow[dddd, "i"'] & & & & A \times [0,1] \arrow[dl, "g \times \mathrm{Id}"'] \arrow[dddd] \arrow[d, phantom, "(2)"] \\
& B \arrow[rr] \arrow[dd] & & B \times [0,1] \arrow[dl, "h"'] \arrow[dd] & \\
& & Y & & \\
& B \sqcup_A X \arrow[rr] \arrow[ur, "f"] \arrow[dr, phantom, "(1)"] & & (B \sqcup_A X) \times [0,1] \arrow[ul, dashed, "F \cdot (3)"'] & \\
X \arrow[ur] \arrow[rrrr] & & & & X \times [0,1] \arrow[ul]
\end{tikzcd}
$$

(3) is a pushout (colimit) of (1) and (2)

$$(B \sqcup_A X) \times [0,1] = (B \times [0,1]) \sqcup_{A \times [0,1]} (X \times [0,1])$$

$\square$

The upshot: If (X, Z) is a cell pair then for a generalised (co)homology theory E,

$$E_n(X, Z) \cong \tilde{E}_n(X/Z)$$

$$E^n(X, Z) \cong \tilde{E}^n(X/Z).$$