Note Title

MATH 695

9/14/2022

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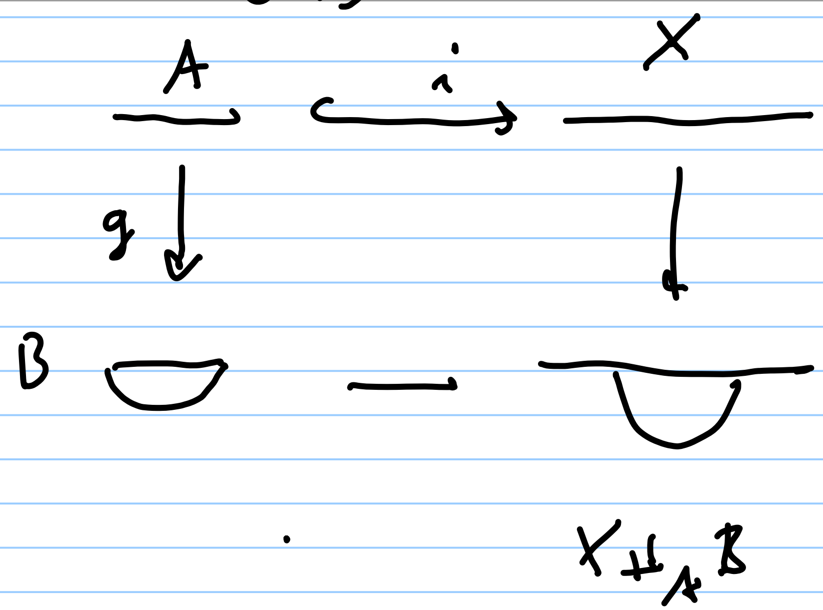
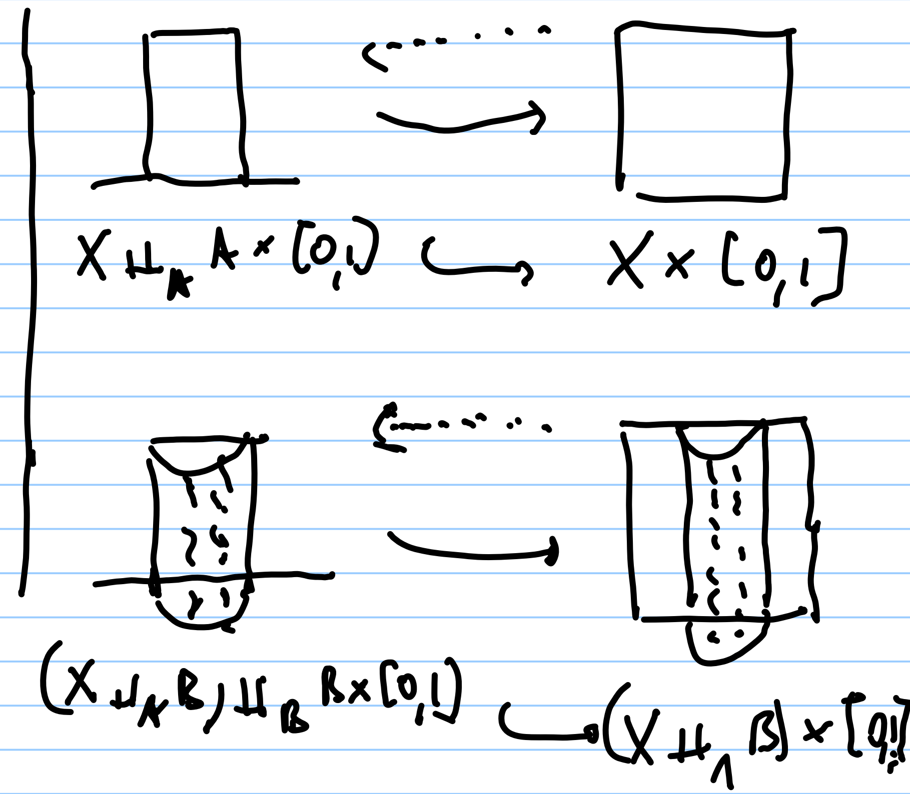
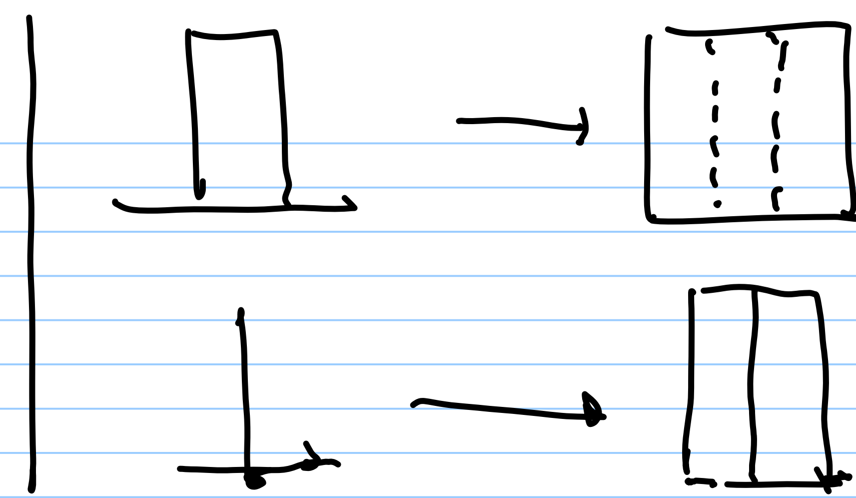




eg does not have to be injective

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ \ast \arrow[r] & X/A \end{tikzcd}$$



Based vs. unbased axioms of (generalized) (co) homology

Eilenberg – Steenrod axioms (calculate (co)homology) of a CW-cx.)

Phase for CW-complexes

$\tilde{E}_n$: k-based CW-cxes $\longrightarrow$ Ab

$\left(\tilde{E}^n : k\text{-based CW-cxes}^{op} \to \text{Ab} \right)$

① (X, Y) based CW-pair, $i: Y \to X$, $j: X \to X/Y$, we have an exact sequence:

$$\tilde{E}_n Y \xrightarrow{\tilde{E}_n i} \tilde{E}_n X \xrightarrow{\tilde{E}_n j} \tilde{E}_n X/Y$$

$$\left(\tilde{E}^n Y \xleftarrow{\tilde{E}^n i} \tilde{E}^n X \xleftarrow{\tilde{E}^n j} \tilde{E}^n X/Y \right)$$

② Given a natural isomorphism

$\sigma: \tilde{E}_n X \cong \tilde{E}_{n+1} \Sigma X$ based isomorphism

$(\sigma: \tilde{E}^n X \cong \tilde{E}^{n+1} \Sigma X)$

③ Infinite arrows: $\tilde{E}_n \bigvee_{i \in I} X_i \cong \bigoplus_{i \in I} \tilde{E}_n X_i \ \cdots$

$\coprod_i X_i / *_i \sim *_j$

$f: Y \to X$ is a based map

$C_{\text{based}} f = Cf / * \times [0,1]$

based

$i: X \to C_{\text{based}} f$

unbased

$Y \times [0,1]$

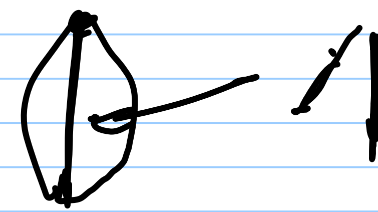
X

$* \times [0,1]$

$$\Sigma X = C_{\text{based}}(X \to *)$$

based suspension

$$= X \times [0,1] \Big/ \begin{array}{l} (x,0) \sim (x',0) \\ (y,1) \sim (y',1) \\ (x,t) \sim (x,t') \end{array}$$

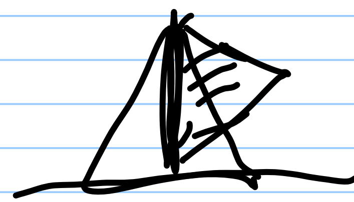


If $Y \xhookrightarrow{i} X$ is a CW-pair $C_{\text{based}}\, i \simeq C i$

$$C_{\text{based}}\, i = Ci \,/\, \underset{\text{IL}}{*} \times [0,1]$$

$C_j \simeq C_i$ $\quad j: I \times [0,1] \to C_i$

C_i is a deformation retract of C_j



In particular, for X a based CW-complex,

based suspension $\to$ $\Sigma X \simeq SX$ unbased suspension

Theorem: As long as we are in the category of CW-complexes, the based and unbased $\circledast$ Eilenberg-Steenrod axioms are equivalent.

Proof start: Given $\tilde{E}_n$, (X,Y) = CW-pair, define

$$E_n(X,Y) := \tilde{E}_n(X/Y).$$

$$E_n(X) := \tilde{E}_n(X_+)$$

$X_+ = X \sqcup \{*\}$

X with a disjoint base point

(similarly for cohomology)

Given $E_n(X)$, $E_n(X,Y)$, define for X based

$$\tilde{E}_n(X) = E_n(X, \{*\}).$$

$\square$

$Y \subseteq X$ CW-pair

$Z \subset Y$ $\bar{Z} \subseteq$ Int Y

$X \setminus Z / Y \setminus Z \cong X/Y$.

Brown's axioms are simpler, can be used to construct examples of generalized cohomology theories.

(6) homology of a CW-complex.

Let X be a CW-complex. We have the skeleta X_n. For ordinary homology with coefficients in A, we have

$$H_n(X_n, X_{n-1}; A) \cong \tilde{H}_n(\underbrace{X_n/X_{n-1}}_{\bigvee S^n}; A) = \bigoplus_{I_n} A$$

set of n-cells $\to I_n$

$$= A I_n$$
$$= A \otimes \mathbb{Z} I_n$$

$$
\begin{tikzcd}
{} \arrow[d] \arrow[dd, bend right=40, "d^{cell} = d^{cell}_n"'] & \\
H_{n-1}(X_{n-1};A) \arrow[d, "d"] \arrow[dd, bend left=60, "0"] & \\
H_{n-1}(X_{n-1},X_{n-2};A) = A I_{n-1} \arrow[d, "\partial"'] \arrow[dd, bend right=40, "d^{cell}_{n-1}"'] & \\
H_{n-2}(X_{n-2};A) \arrow[d] & \\
H_{n-2}(X_{n-2},X_{n-3};A) &
\end{tikzcd}
$$

$$\tilde{H}_k(S^n;A) = \begin{cases} A & k = n \\ 0 & k \neq n \end{cases}$$

$S^n = \underbrace{S \cdots S}_{n\text{-fold suspension}} S^0$

$X_i = \emptyset$

$i < 0$

<u>Summary:</u> Let X be a CW-complex

$C^{cell}(X;A)$:

$$\begin{tikzcd}
{} \arrow[d] & {} \arrow[d] \\
C^{cell}_n(X;A) \arrow[d, "d^{cell}"'] & = & \mathbb{Z}I_n \otimes A \arrow[d, "d^{cell}_{\mathbb{Z}} \otimes A"] \\
C^{cell}_{n-1}(X;A) \arrow[d] & = & \mathbb{Z}I_{n-1} \otimes A \arrow[d] \\
{} & & {}
\end{tikzcd}$$

$$H^{cell}_n(X;A) := H_n(C^{cell}(X;A))$$

$$C^*_{cell}(X;A) := \mathrm{Hom}\left(C^{cell}(X;\mathbb{Z}), A\right)$$

$$H^n_{cell}(X;A) := H^n\left(C^*_{cell}(X;A)\right)$$

<u>Theorem</u>: $$H^{cell}_n(X;A) \cong H_n(X;A)$$

$$H^n_{cell}(X;A) \cong H^n(X;A).$$

HW 2: $$\mathbb{CP}^n = \{(z_0, \dots z_n) \in \mathbb{C}^{n+1} \mid \textstyle\sum |z_k|^2 = 1\} \Big/ \, S^1$$ S^{2n+1}

$/(z_0, \dots z_n)$

$\sim (\lambda z_0, \dots \lambda z_n)$

$\lambda \in S^1 = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$

(a) $\mathbb{CP}^n$ is a CW-complex

[Hint: $2k$-skeleton $= (2k+1)$-skeleton $= \mathbb{CP}^k$]

$\mathbb{CP}^k \subset \mathbb{CP}^n$

$(z_0, \dots z_k) \mapsto (z_0, \dots z_k, 0 \dots 0)$.

(b) Calculate $H_k(\mathbb{CP}^n; A)$, $H^k(\mathbb{CP}^n; A)$ using cell (CW) homology.

This can be done because the differential is 0. But that is not always the case.

So how do we compute the CW-differential?

$$\mathbb{Z}I_n \xrightarrow{d} \mathbb{Z}I_{n-1}$$

	I_n
I_{n-1}	

given $i \in I_n$ $j \in I_{n-1}$, produce a number.

columns are finite. (all entries except finitely many $= 0$)

$$\begin{array}{ccc} D^n & \xrightarrow{i\text{ cell}} & X_n \\ \cup & & \cup \\ S^{n-1} & \xrightarrow{\varphi_n} & X_{n-1} \end{array} \longrightarrow X_{n-1}/X_{n-2} = \bigvee_{\nu \in I_{n-1}} S^{n-1} \longrightarrow S^{n-1}$$

$\varphi_n|_{S^{n-1} \times \{i\}}$

send all $\bigvee$ summands to $*$ <u>except</u> $\nu = j$

We have produced a map

$$S^{n-1} \xrightarrow{\varphi} S^{n-1}$$

The coefficient of the matrix is whatever this map induces in $\underline{\underline{H_{n-1}}}$.

$$H_{n-1}(S^{n-1}) = \mathbb{Z} \qquad \text{(slight disc needed for } n=1)$$

This is the <u>degree</u> $\deg(\varphi)$.