Note Title 9/16/2022

MATH 695 9/16/2022

$C^{\text{chain}}(X) : \quad \longrightarrow \mathbb{Z}\, I_n \xrightarrow{d_n^{\text{cell}}} \mathbb{Z}\, I_{n-1} \longrightarrow$

CW-complex $\quad$ n-cells

$d_1^{\text{cell}} : \mathbb{Z}\{1\text{-cells}\} \longrightarrow \mathbb{Z}\{0\text{-cells}\}$

1-cell represents a map

$e_i : [0,1] \longrightarrow X_1$

$d_1^{\text{cell}}(e_i) = e_i(0) - e_i(1)$

$n > 1$

$$\begin{tikzcd} e_i : D^n \arrow[r] & X_n & & \\ \cup & \cup\mid & & \\ S^{n-1} \arrow[r, "\text{attaching map}"'] \arrow[rrr, bend right] & X_{n-1} \arrow[r] & X_{n-1}/X_{n-2} \arrow[r] & S^{n-1} \cdot \{y\} \end{tikzcd}$$

collapse all cells but one

The theory of maps $S^k \to S^k$:

Theorem (Hopf): If M is a compact connected oriented k-manifold, then the set of homotopy classes of maps

$M \to S^k$
is canonically bijective with $\mathbb{Z}$ via a map called
__degree__:

$$\deg : [M, S^k] \xrightarrow{\cong} \mathbb{Z}$$

[Details: Milnor: Topology from a differential viewpoint]

__Proof sketch__ (how to calculate degree geometrically):
Let $f : M \to S^k$ be a map.

Step 1 : Approximate f by a smooth map (e.g. $f_n \rightrightarrows f$)

smooth

Then $\exists\ x \in S^k$ such that $f^{-1}(x)$ is finite

and for $y \in f^{-1}(x)$: $Df_y : TM_y \xrightarrow{\cong} TS^k_x$

tangent space.

(In fact, you can define the regular value even when f is not smooth everywhere.)

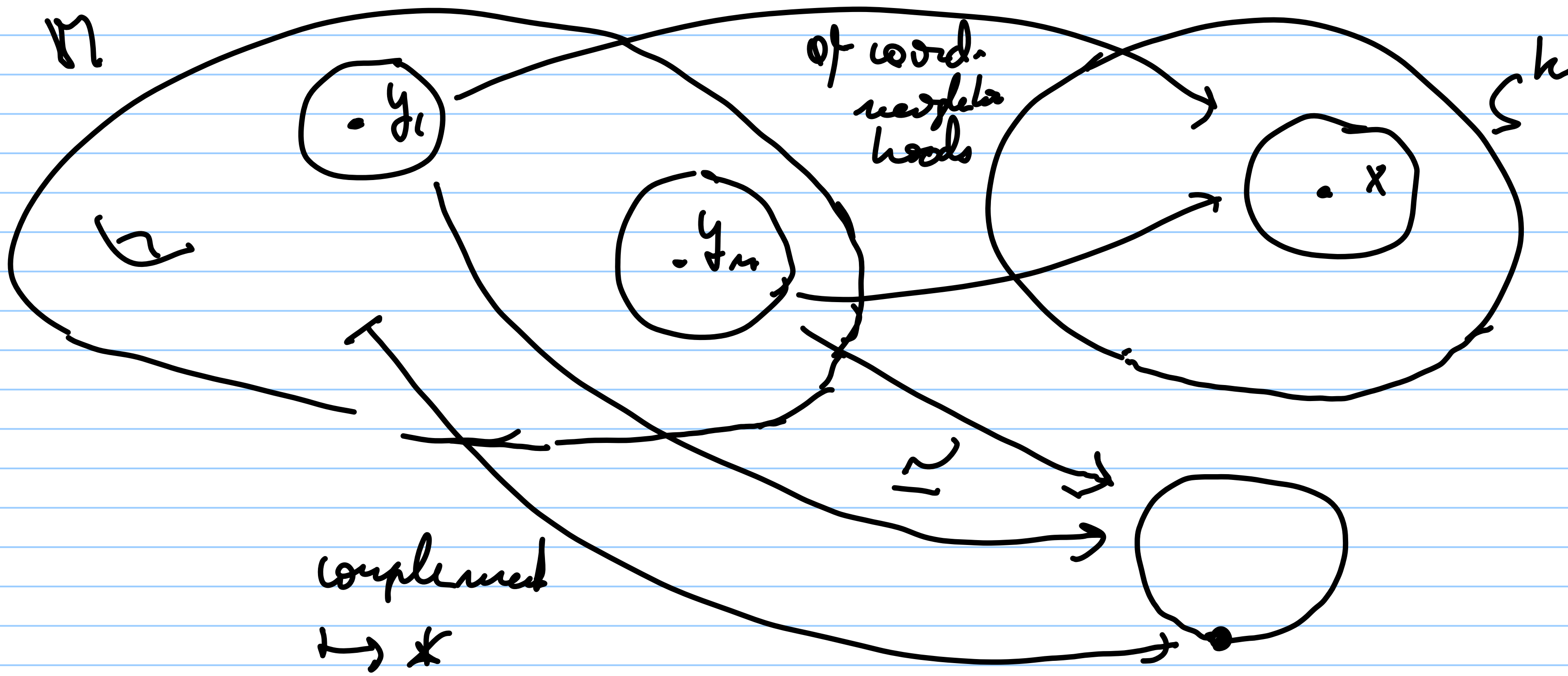
If x is a regular value of f then

$$\deg f = \sum_{y \in f^{-1}(x)} \operatorname{sign}(\det Df_y)$$

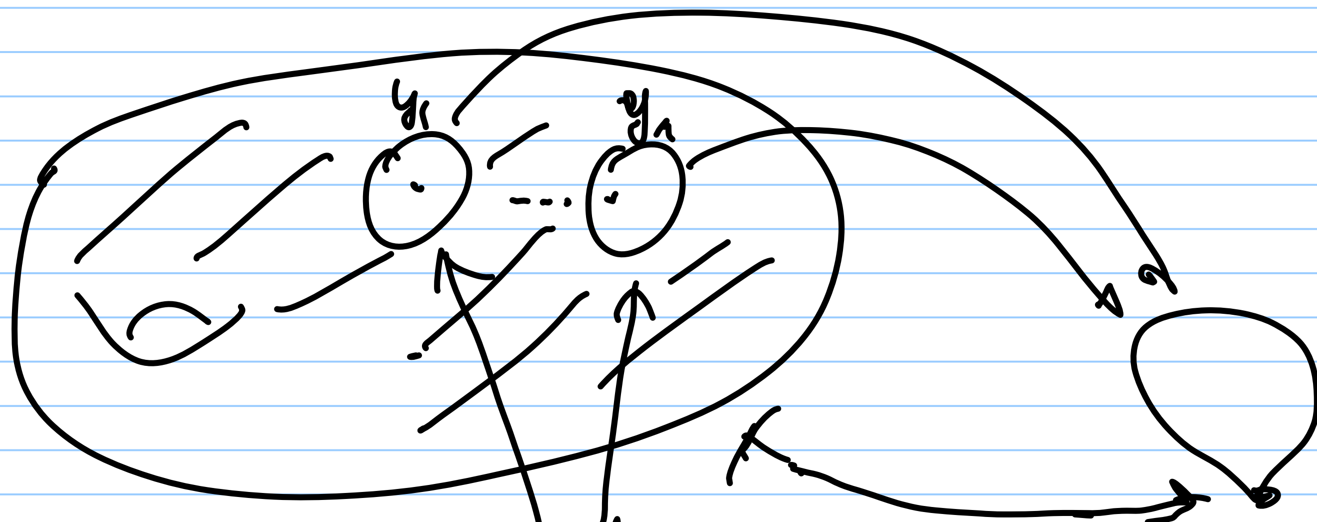
this depends on coordinates $t_1, \dots t_k$ at y,
choose them so that
$\frac{\partial}{\partial t_1}, \dots \frac{\partial}{\partial t_k}$
has positive orientation.
(alternatively, if ω is a nowhere 0 k-form,
$(dt_1 \wedge \dots \wedge dt_k)_y = \alpha\, \omega_y$
$\alpha > 0$).

S^k is oriented,
choose orientation
once and for all.

To prove Hopf's thm, dfn. $\{y_1 \ldots y_n\} = f^{-1}(x)$

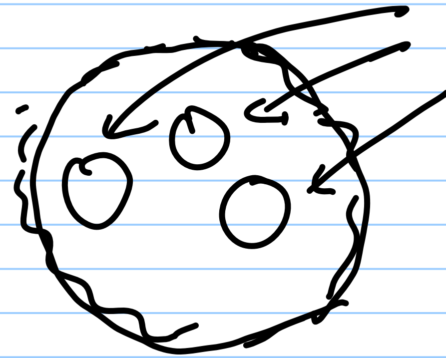


To prove that maps of the $\underbrace{\text{same degree}}_{\substack{\text{assume at the}\\ \text{same reg. value.}}}$ are homotopic.



move those point to the same coord. neighborhood

Then WOLOG, $M = S^k = D^k / S^{k-1}$

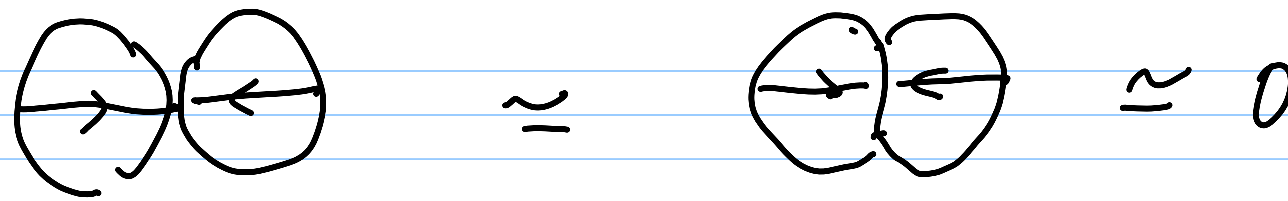


these maps can be assumed linear

$GL_k(\mathbb{R})$ has two path-components which are determined by sign of det.

Can assume the maps are

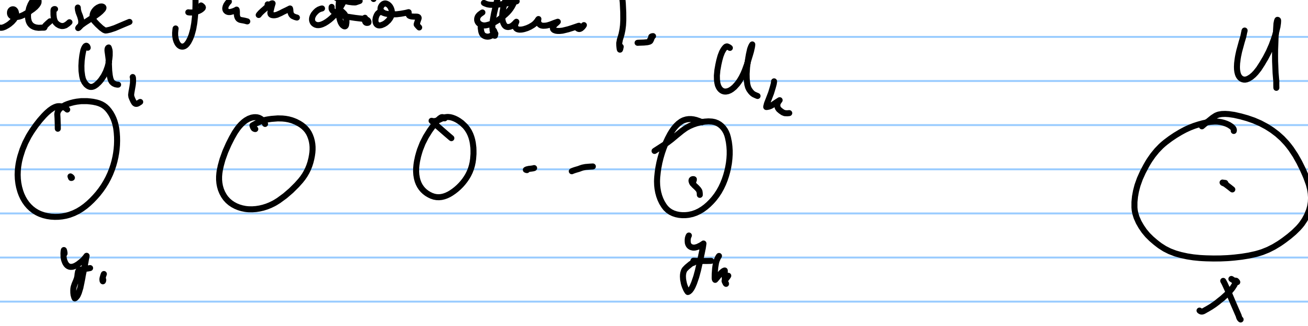
$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \text{ or } \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$



To show that homotopic maps have the same degree: Method 1 (cheat): $H_k(M) = \mathbb{Z}$ (and the above maps induce multiplication by the degree in homology)

A <u>geometric proof</u> involves a local component and a global component.

<u>local</u> : The choice of x within a small neighborhood does not matter.
(inverse function thm).



$f: U_i \xrightarrow{\cong} U$ diffeomorphically if x is a regular value of f.

global (also addresses global choice of x):

$h: f \simeq g : M \to S^k$

$h: M \times [0,1] \to S^k$ } smooth approximate the homotopy

$h_n \rightrightarrows h$

h_n smooth

$\exists$ dense set of $x \in S^k$ such that

x is a regular value of h

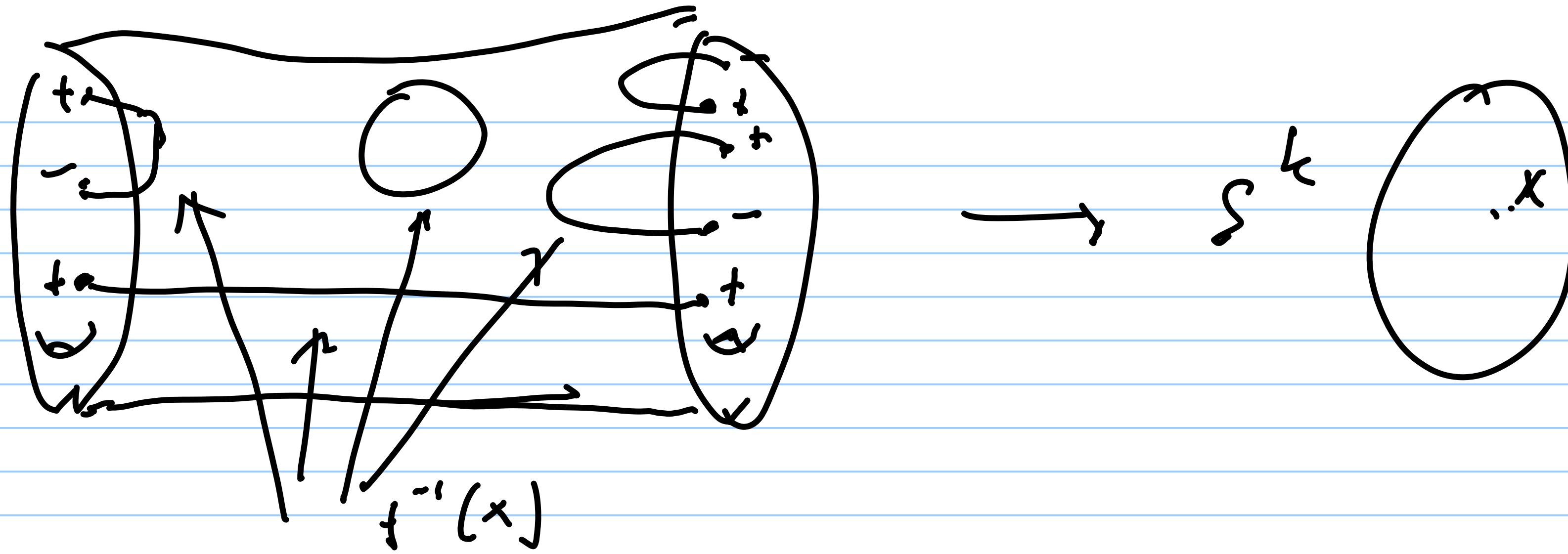
(this is where we need the local argument)

$\forall y \quad h(y) = x \qquad Dh : TM \times [0,1]_y \to TS^k_y$

is onto.

$h^{-1}(x)$ is a compact smooth 1-manifold w. boundary.

HW3 Milnor proves: Every compact smooth connected 1-mfld is $\underset{\text{diffeomorphic}}{\cong}$ S^1 or $[0,1]$



$\square$

Example: $X = \mathbb{RP}^n$. Calculate its homology.

The cell filtration: $= \{ \underbrace{(x_0, \ldots x_n) \in \mathbb{R}^{n+1} / \sum x_i^2 = 1}\} /$

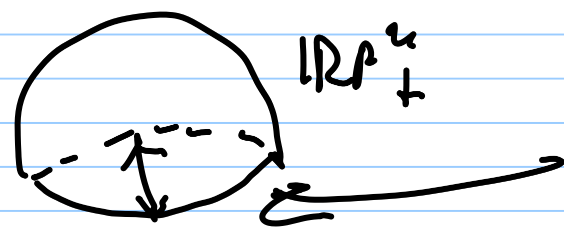
$(\cdot \subseteq \ldots \subseteq \ldots)$

$\mathbb{RP}^0 \subseteq \mathbb{RP}^1 \subseteq \cdots \subseteq \mathbb{RP}^n \qquad S^n \qquad (x_0, \dots, x_n) \sim (-x_0, \dots, -x_n)$

make the last coordinate 0.

n-cell

$\mathbb{RP}^n_+ = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \sum x_i^2 = 1,\ x_n \geq 0\} \cong D^n$

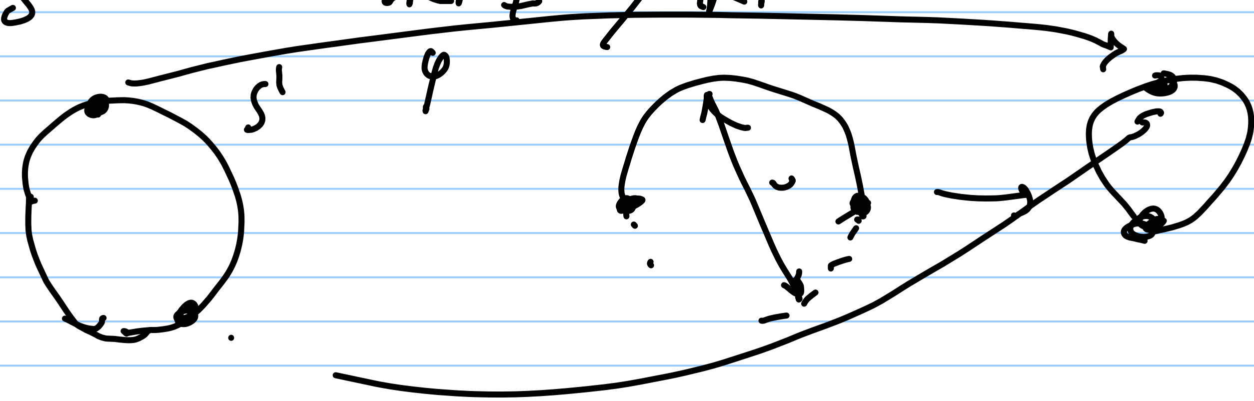


$\mathbb{RP}^n_+$

Boundary: $x_n = 0$

$\cong S^{n-1} \xrightarrow[\text{projection}]{} \mathbb{RP}^{n-1}$

$S^{n-1} \xrightarrow{\text{hori}} \mathbb{R}P^{n-1}_{+} / \mathbb{R}P^{n-2}$



sign alternates depending on n

$$\deg(\varphi: S^{n-1} \to S^{n-1}) = \begin{cases} 2 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

The

$C^{cell}(\mathbb{RP}^n)$;

$$\begin{array}{lccccccccccc}
 & \mathbb{Z} & \xrightarrow{1+(-1)^n} & \mathbb{Z} & \cdots & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} \\
\text{degree} & n & & n-1 & & & & & & 1 & & 0
\end{array}$$

$$H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & \text{if } k=0 \\ \mathbb{Z}/2 & \text{if } k<n \text{ odd} \end{cases}$$

$;\mathbb{Z}$

$\mathbb{Z}$ if $k = n$ odd