Note Title

MATH 695

9/19/2022

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HW 1: Given $S^3 = \{(z,t) \in \mathbb{C}^2 \mid |z|^2 + |t|^2 = 1\}$ and $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ acts by

$$\lambda(z,t) = (\lambda z, \lambda t).$$

Consider $\mathbb{Z}/k \subset S^1$ given by the kth root of 1. $e^{2\pi i m/k}$.

Let $L_k = S^3/(\mathbb{Z}/k)$. Compute

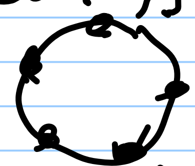
$$H_*(L_k;\mathbb{Z}), \quad H^*(L_k;\mathbb{Z}).$$

[Hint: Find a CW-structure by finding a CW-structure on S^3 such that $\mathbb{Z}/k$ acts on the sets of cells compatibly with the $\mathbb{Z}/k$-action given above and in each open cell $\cong (D^n \setminus S^{n-1})$ every point x has the same isotropy $\{g \in \mathbb{Z}/k \mid gx = x\}$.

$\mathbb{Z}/k$ - equivariant CW-cx.

not needed here - $\mathbb{Z}/k$ acts freely $\Rightarrow$ no topology is trivial. Start with

$$S^1 \subset S^3$$
$$\{(z,0)\} \quad \{(z,t)\}$$



0-cells

also need 2-cells & 3-cells).

We established CW-homology $\boxed{}$.

(cw)

What happens in generalised (co)homology E?

Recall that

k-fold suspension

$$\tilde{E}_n(S^k) = \tilde{E}_{n-k}(S^0) = E_{n-k}(*)$$

$$= E_{n-k}$$

notation

called coefficients groups

Let X be a CW-complex.

$$\tilde{E}_n(X_k/X_{k-1}) = E_n(X_k, X_{k-1}) \xrightarrow{\partial} E_{n-1}(X_{k-1})$$

$$\tilde{E}_n(X_k/X_{k-1}) \cong \bigoplus_{i \in I_k} E_{n-k} = E_{n-k} \cdot I_k$$

$$E_{n-1}(X_{k-1}) \to E_{n-1}(X_{k-1}, X_{k-2}) = \tilde{E}_{n-1}(X_{k-1}/X_{k-2})$$

$$\tilde{E}_{n-1}(X_{k-1}/X_{k-2}) \cong \bigoplus_{j \in I_{k-1}} E_{n-k} = E_{n-k} \cdot I_{k-1}$$

We get, for each $q \in \mathbb{Z}$ a chain complex

$$\begin{array}{ccccl}
\to E_q \cdot I_k & \xrightarrow{E_p \otimes d^{\text{cell}}} & E_q \cdot I_{k-1} & \to & \\
\shortparallel & & \shortparallel & & \\
E_q \otimes \mathbb{Z} I_k & & E_q \otimes \mathbb{Z} I_{k-1} & &
\end{array}$$

$\Big\}$ $\mathbb{Z}$-graded chain complex.

$\underbrace{}$ $\mathbb{Z}$-indexed system of chain cxes.

The additional grading makes it doubtful that this can calculate $E_n X$. should be related to "$E_{n+k} X$".

Instead, we get a spectral sequence.

can be homological or cohomological

Let's talk about homological first.

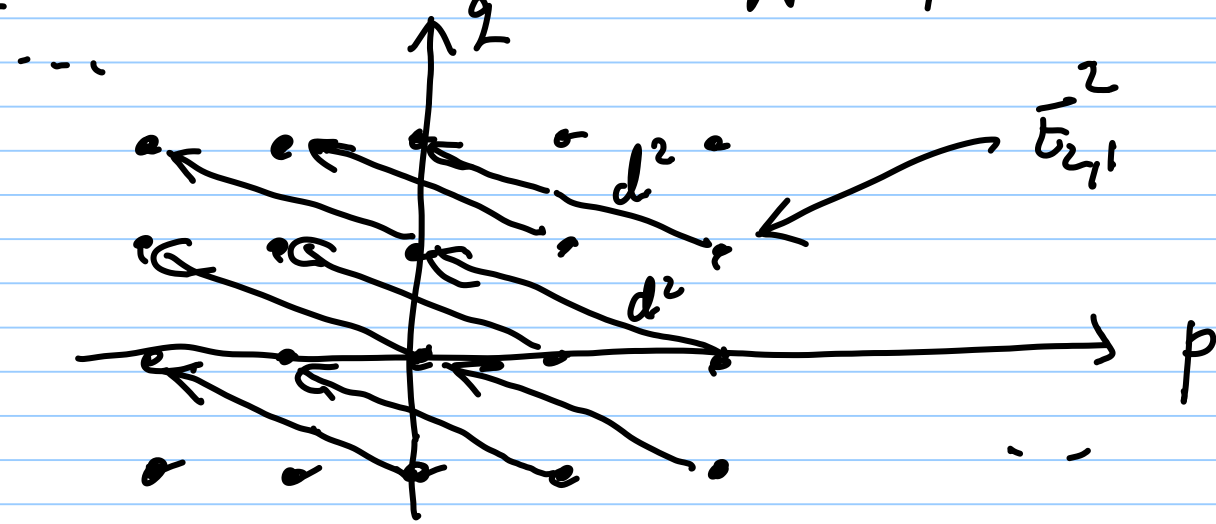
not related to the Gahove

A homological spectral sequence $E^{\cdot}$

has an r-page E^r_{pq} consisting of ab. groups E^r_{pq}.

and differentials $d^r : E^r_{pq} \longrightarrow E^r_{p-r,\,q+r-1}$ for $r \geq r_0$ (usually, $r_0 = (0), 1, 2$) such that $d^r d^r = 0$, and E^{r+1}_{pq} = the homology of d^r at bidegree p,q.

<u>Example</u>: the 2-page:

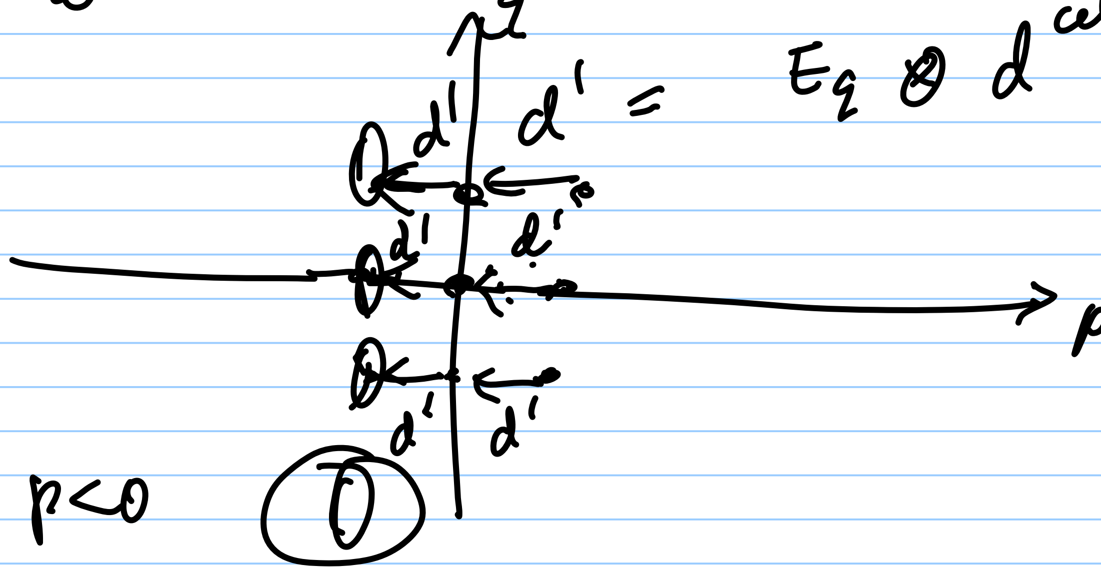


Example: In our case of a generalised homology theory,

generalised homology coeff.

spectral sequence $E^1_{pq} = E_q \otimes \mathbb{Z}I_p$

$d^1 = E_q \otimes d^{cell}$



$p<0$

$E_q = E_q(*)$

Therefore, we can calculate E^2_{pq}. (We do not necessarily know $d^2, d^3, \ldots$.) $E^2_{pq} = H^{cell}_p(X; E_q)$

Assume we know the higher differentials.
What can we conclude?
A spectral sequence E^r_{pq} is called <u>convergent</u> if $\forall p,q \in \mathbb{Z}\ \exists R\ \forall r > R\ \ d^r\big|_{E^r_{pq}} = 0$.

Then we can define $E^{\infty}_{pq} = \operatorname{colim}_r E^r_{pq}$.

Usually, one hopes that E^{∞}_{pq} is related to what we are trying to calculate, typically as an <u>associated graded object</u>.

In the example of generalized homology of a CW-complex, we have the following

<u>Theorem</u> (Atiyah-Hirzebruch spectral sequence):
Let X be a CW-complex and let E be a generalized homology theory. Then there exist a natural (on pages $r \geq 2$) spectral
(under continuous maps)
sequence with

$$E^2_{pq} = H_p^{cell}(X; E_q)$$

such that

associated graded

$$E^{\infty}_{pq} = F_p E_{p+q}(X) \Big/ F_{p-1} E_{p+q}(X)$$

where

$$F_p E_n X = \mathrm{Im}\left(E_n(X_p) \xrightarrow{E_n \subseteq} E_n(X) \right).$$

convergent filtration

increasing system of subgroups
whose union is $E_n(X)$.

We write:

$$E^2_{pq} = H^{cell}_p(X; E_q) \Rightarrow E_{p+q}(X)$$

Definition: A (increasing) <u>convergent filtration</u> of an abelian group G is a sequence of subgroups

$$\cdots \subseteq F_{p-1}G \subseteq F_pG \subseteq \cdots$$

$\exists N$ $F_pG = 0$ for $p < N$ (in the example,

$N=0)$

$\bigcup F_p G = G.$

The associated graded abelian group is

$(E^0 G)$ where $(E^0 G)_p = F_p G / F_{p-1} G.$

Next time: ① AHSS $\Rightarrow$ CW homology thm.

$(H^{cell} = H)$

② Proof of AHSS (how to construct the spectral sequence?)