MATH 695 Note Title 9/21/2022 9/21/2022

Spectral sequences

Where does a spectral sequence come from?

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

D, E abelian groups (typically, bi-graded) $\mathbb{Z} \times \mathbb{Z}$

<u>exact</u>: $\operatorname{Ker} j = \operatorname{Im} i$

$\operatorname{Ker} k = \operatorname{Im} j$, $\operatorname{Ker} i = \operatorname{Im} k$.

= Exact couple

Philosophy: Try to get information on D from knowing E.

The kind of information we are entitled to is precisely a *spectral sequence*.

Example: X CW-complex, E generalized homology theory

LES

$$\begin{tikzcd} E_{p+q}(X_{p-1}) \arrow[r, "i"] & E_{p+q}(X_p) \arrow[dl, "j'"] \\ E_{p+q-1}(X_{p-1}) & \end{tikzcd} \quad \Bigg\{ \quad \begin{tikzcd} D \arrow[r, "i"] & D \arrow[d, "j"] \\ \arrow[u, "k"] & {} \end{tikzcd}$$

$$\overset{k}{\nwarrow} \quad E_{p+q}(X_p, \overset{\swarrow}{X}_{p-1}) \qquad \overset{\text{E}}{\longleftrightarrow} \text{ adding up over } p,q$$

The bigrading: $E^1_{pq} = E_{p+q}(X_p, X_{p-1})$

$$\left(E_q \quad I_p \right)$$

$$\begin{tikzcd} E \arrow[rr, "{(1,-1)}"] & & E \arrow[dl, "j", "{(0,0)}"'] \\ & D \arrow[ul, "k", "{(-1,0)}"'] & \end{tikzcd}$$

$D^1_{pq} = E_{p+q}(X_p)$

$\boxed{\text{bidegree of map: what does it add to } p,q\text{?}}$

Single grading $n = p+q$ — total degree

p — filtration degree

q — complementary degree

In terms of the single grading:

$$\begin{tikzcd} D \arrow[rr, "i", "0"'] & & D \arrow[dl, "j", "0"'] \\ & E \arrow[ul, "k", "-1"'] & \end{tikzcd}$$

Massey observed that we have, given an exact couple, a <u>derived exact couple</u>

given:

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd} \qquad \Bigg| \qquad \begin{tikzcd} D' \arrow[rr, "i'"] & & D' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd}$$

$D' = \operatorname{Im} i$

$E' = \operatorname{Ker} d / \operatorname{Im} d$

where $d = j \circ k$

$i' = i|\partial'$

$k'([x]) = [k(x)]$ ←

$dd = j\circ k\circ \underbrace{j\circ k}_{0} = 0$

supposing $x \in \mathrm{Ker}(d)$
$= \mathrm{Ker}(jk)$

$jk(x) = 0$

$k(x) \in \mathrm{Im}(i)$.

$\boxed{\begin{array}{l} i', k' \text{ induced} \\ \text{by } i, k \\ \text{“} j' = j \circ i^{-1} \text{”} \end{array}}$

What about if $x \in \mathrm{Im}(d)$?

$x = jk(y)$

$k(x) = kjk(y) = 0$.

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

$j'(i(x)) = [j(x)]$

$D' = \mathrm{Im}\, i$

Need to show

$\underbrace{jk}_{0} j(x) = 0$

but $kj(x) = 0$ OK.

Also need: $I_f \; i(x) = 0 \Rightarrow [j(x)] = 0$

$x = k(y) \quad j(x) = \underbrace{jk}(y) = d(y).$

HW 2: Show that

$$\begin{tikzcd} D' \arrow[rr, "i'"] & & D' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd}$$

is an exact couple.

This is called the <u>derived</u> exact couple.

In single grading the degrees are

$$\begin{tikzcd} D \arrow[rr, "0", "i"'] & & D \arrow[dl, "j"', "0"] \\ & E \arrow[ul, "k"', "-1"] & \end{tikzcd} \qquad \Bigg| \qquad \begin{tikzcd} D' \arrow[rr, "0", "i'"'] & & D' \arrow[dl, "j'"', "0"] \\ & E' \arrow[ul, "k'"', "-1"] & \end{tikzcd}$$

i', k' induced by i, k

"$j' = j \circ i^{-1}$"

Bigrading in derived exact couples

$$\begin{tikzcd} D \arrow[rr, "{(1,-1)}", "i"'] & & D \arrow[dl, "j", "{(0,0)}"'] \\ & E \arrow[ul, "k", "{(-1,0)}"'] & \end{tikzcd} \quad \Bigg| \quad \begin{tikzcd} D' \arrow[rr, "{(1,-1)}", "i'"'] & & D' \arrow[dl, "j'", "{(-1,1)}"'] \\ & E' \arrow[ul, "k'", "{(-1,0)}"'] & \end{tikzcd} \quad \Bigg| \longrightarrow \quad \begin{tikzcd} D'' \arrow[rr, "{(1,-1)}", "i''"'] & & D'' \arrow[dl, "j''", "{(-2,2)}"'] \\ & E'' \arrow[ul, "k''", "{(-1,0)}"'] & \end{tikzcd} \quad \cdots$$

Convention $\qquad\qquad\qquad\qquad\qquad \parallel$

$\parallel$

$$\begin{tikzcd} D^1 \arrow[rr, "i^1"] & & D^1 \arrow[dl, "j^1"] \\ & E^1 \arrow[ul, "k^1"] & \end{tikzcd} \qquad \begin{tikzcd} D^2 \arrow[rr, "{(1,-1)}", "i^2"'] & & D^2 \arrow[dl, "j^2", "{(-1,1)}"'] \\ & E^2 \arrow[ul, "k^2", "{(-1,0)}"'] & \end{tikzcd} \qquad \begin{tikzcd} D^3 \arrow[rr, "{(1,-1)}"] & & D^3 \arrow[dl, "{(-2,2)}"'] \\ & E^3 \arrow[ul, "{(-1,0)}"'] & \end{tikzcd}$$

$d^1 = j^1 \circ k^1$ has degree $(-1,0)$

$d^2 = j^2 \circ k^2$ has degree $(-2,1)$

$d^3 = j^3 \circ k^3$ degree $(-3,2)$

$$\begin{tikzcd}
D^r \arrow[rr, "{(1,-1)}", "i^r"'] & & D^r \arrow[dl, "j^r"', "{(1-r,\, r-1)}"] \\
& E^r \arrow[ul, "k^r"', "{(-1,0)}"] &
\end{tikzcd}$$

$$\boxed{\begin{gathered} d^r = j^r \circ k^r \\ \text{has degree} \\ (-r, r-1) \\ E^{r+1} = H(E^r, d^r) \end{gathered}}$$

a homological spectral sequence!

We also need a direct description of the $(r-1)$st derived ~~exact~~ couple

$$\begin{tikzcd} D^r \arrow[rr] & & D^r \arrow[dl] \\ & E^r \arrow[ul] & \end{tikzcd}$$

The start is an direct description of E', or the alternative derived exact couple, avoiding d:

$$E' = k^{-1}(\mathrm{Im}\, i) / j(\mathrm{Ker}\, i)$$

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

Why $\underbrace{j(\mathrm{Ker}\, i) \subseteq k^{-1}\mathrm{Im}\, i}_{\Updownarrow}$

$\underbrace{kj(\mathrm{Ker}\, i)}_{= 0} \subseteq \mathrm{Im}\, i$ OK.

Why $k^{-1}(\mathrm{Im}\, i) / j(\mathrm{Ker}\, i) \cong \mathrm{Ker}(j \circ k) / \mathrm{Im}(j \circ k)$

$$k^{-1}(\operatorname{Im} i) = k^{-1}(\operatorname{Ker} j) = \operatorname{Ker}(j \circ k)$$

$$j(\operatorname{Ker} i) = j(\operatorname{Im} k) = \operatorname{Im}(jk) \checkmark$$

This generalizes. By induction, if we denote the nth derived exact couple by

$$\begin{tikzcd} D^{(n)} \arrow[rr, "i^{(n)}"] & & D^{(n)} \arrow[dl, "j^{(n)}"] \\ & E^{(n)} \arrow[ul, "k^{(n)}"] & \end{tikzcd}$$

$$D^{(n)} = \operatorname{Im}(i \circ \underbrace{0 \cdots 0}_{n} \circ i)$$

$$E^{(n)} = k^{-1}\operatorname{Im}(i \circ \underbrace{0 \cdots 0}_{n} \circ i)$$

$j^* \ker(\underbrace{i^* 0 \cdots 0 i^*}_{r})$.

In the other notation $E^r = E^{(r-1)}$, $D^r = D^{(r-1)}$:

$$D^r = \mathrm{Im}(\underbrace{i^* 0 \cdots 0 i^*}_{r-1})$$

$$\boxed{E^r = h^{-1}\,\mathrm{Im}(\underbrace{i^* 0 \cdots 0 i^*}_{r-1}) \,/\, j^* \ker(\underbrace{i^* 0 \cdots 0 i^*}_{r-1}).}$$