MATH 695 9/23/2022

Note Title 9/23/2022

$\forall p,q \in \mathbb{Z}\ \exists R\ \forall r \geq R\ \left(0 = d^r : E^r_{pq} \to E^r_{p-r,q+r-1}\right)$.

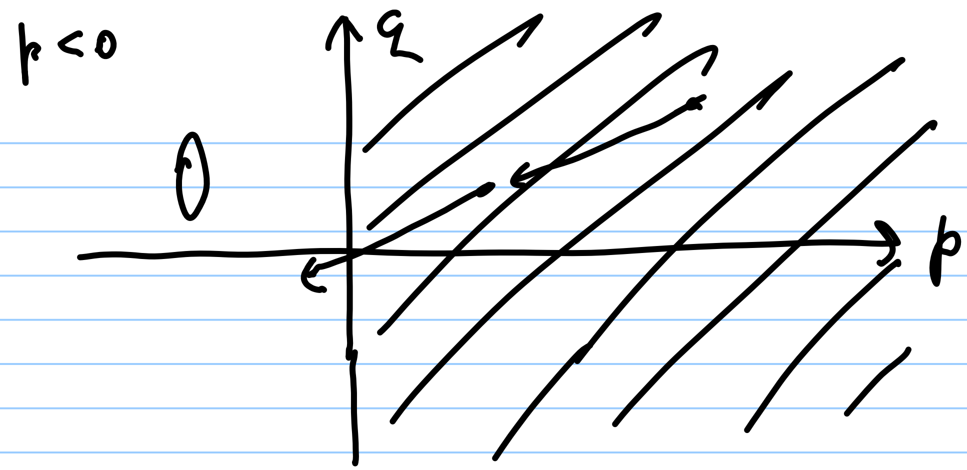
convergence

$$E^{r+1}_{pq} = \underbrace{\operatorname{Ker} d^r|_{E^r_{pq}}}_{Z^r_{pq}} \Big/ \underbrace{\operatorname{Im} d^r : E^r_{p+r,q-r+1}}_{B^r_{pq}}$$

$E^\infty_{pq} = \operatorname{colim} E^r_{pq}$

Example:

AHSS in E_* (generalized homology)



More about convergence: Recall

$$D^r = \operatorname{Im} i^{r-1}$$

$$E^r = k^{-1}(\operatorname{Im} i^{r-1}) \big/ j(\operatorname{Ker} i^{r-1})$$

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

exact couple

What guarantees convergence is if for each p,q

$\operatorname{Im}\ \cdots \xrightarrow{i^{-m}} D^{n}_{pq}$

composition

some constant depending on p,q

if it stays constant for $m > M(p,q)$

Fix p,q. The concept, for an infinite system of maps

$$\cdots \longrightarrow \xrightarrow{f_5} \xrightarrow{f_4} A_4 \xrightarrow{f_3} A_3 \xrightarrow{f_2} A_2 \xrightarrow{f_1} A_1$$

If the images of the compositions of the f_i's stay eventually constant at each target A_n

$$A_m \to \cdots \to A_n$$

for $m > N(n)$, we say that the system satisfies the Mittag-Leffler condition.

Example: Recall the problem of an inverse limit of short exact sequences

$$
\begin{tikzcd}
0 \arrow[d] & & & 0 \arrow[d] & 0 \arrow[d] \\
\varprojlim A_n \arrow[d] & \cdots \arrow[r] & & A_2 \arrow[r] \arrow[d] & A_1 \arrow[d] \\
\varprojlim B_n \arrow[d] & \cdots \arrow[r] & & B_2 \arrow[r] \arrow[d] & B_1 \arrow[d] \\
\varprojlim C_n & \cdots \arrow[r] & & C_2 \arrow[r] \arrow[d] & C_1 \arrow[d] \\
 & & & 0 & 0
\end{tikzcd}
$$

This map is onto (lim of an inverse sequence preserves exactness) when the system

$$\cdots \xrightarrow{f_3} A_2 \xrightarrow{f_1} A_1 \qquad \circledast$$

satisfies the Mittag-Leffler condition.

HW ③ Prove that lim of an inverse sequence preserves exactness when the in system $\circledast$ every map f_i is onto.

Back to the spectral sequence: When the Mittag-Leffler condition is satisfied for each D_{pq}, we have

$$E_\infty = \left[k^{-1}\left(\bigcap_m \mathrm{Im}(i^m)\right) \Big/ j\left(\bigcup_m \ker(i^m)\right) \right]$$

Note: this definition can be made in general

at each p,q, this is eventually constant

<u>Example:</u> AHSS

X CW-cx.

Exact couple:

$D^1_{pq} = E_{p+q}(X_p)$

$E^1_{pq} = E_{p+q}(X_p, X_{p-1})$

Mittag-Leffler condition:

$$0 \to E_{p+q}(X_0) \to E_{p+q}(X_{p-1}) \to E_{p+q}(X_p)$$

$$I_n \xrightarrow{i^n} E_{p+q}(X_n) = 0 \text{ if } n > p.$$

$$\quad\quad\quad\quad\quad\quad \cup$$

$$\quad\quad\quad\quad\quad\quad D_{pq} \quad\quad E_{p+q}X = \varinjlim E_{p+q}X_p$$

$$E^{\infty}_{pq} = \underbrace{k^{-1}(0)_{pq}}_{\mathrm{Ker}\,k} \Big/ j(\mathrm{Ker}(E_{p+q}X_p \to E_{p+q}X)) =$$

$\#$

$$
\begin{tikzcd}
D \arrow[r] & D \arrow[dl, "j"] \\
E \arrow[u, "k"] &
\end{tikzcd}
$$

$$k : E_{p+q}(X_p, X_{p+1}) \to E_{p+q-1}(X_{p-1})$$

$$\operatorname{Im} j \left(E_{p+q} X_p \to E_{p+q}(X_p, X_{p-1}) \right)$$

$$\frac{E_{p+q}(X_p)}{\operatorname{Im}\left(E_{p+q} X_{p-1} \to G_{p+q} X_p\right)}$$

$$\|$$

$$j\left(\operatorname{Ke}\left(E_{p+q} X_p \to E_{p+q} X\right)\right) \overset{?}{=} \frac{\operatorname{Im}\left(G_{p+q} X_p \to E_{p+q} X\right)}{\operatorname{Im}\left(E_{p+q} X_{p-1} \to E_{p+q} X\right)}$$

$$\|$$

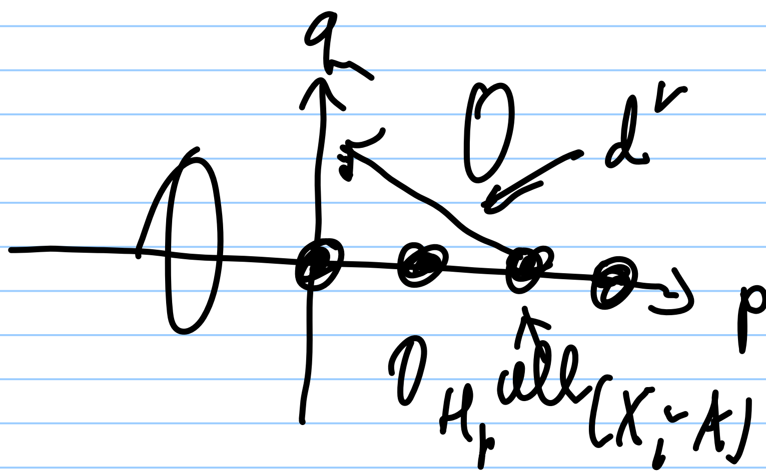
$$\underline{G_{p+q}(X_p)} \Big/ \left(\mathrm{Im}(G_{p+q}X_{p-1} \to G_{p+q}X_p) + \underline{\mathrm{Ker}(G_{p+q}X_p \to G_{p+q}X)}\right)$$

$$= \frac{G_{p+q}(X_p)}{\mathrm{Ker}(G_{p+q}X_p \to G_{p+q}X)} \Big/ \mathrm{Im}\left(G_{p+q}X_{p-1} \to G_{p+q}X_p\right)$$

$$= \mathrm{Im}\left(G_{p+q}X_p \to G_{p+q}X\right) \Big/ \mathrm{Im}\left(G_{p+q}X_{p-1} \to G_{p+q}X\right)$$

How does this apply to $\underbrace{\text{ordinary homology}}_{\text{singular homology with coefficients } A}$?

$r \geq 2$



$$E^2_{pq} = H_p^{cell}(X; H_q(*)), \quad H_q(*) = \begin{cases} A & \text{for } q = 0 \\ 0 & \text{else} \end{cases}$$

$$\Rightarrow H_{p+q}(X)$$

$d^r = 0 \quad r \geq 2$

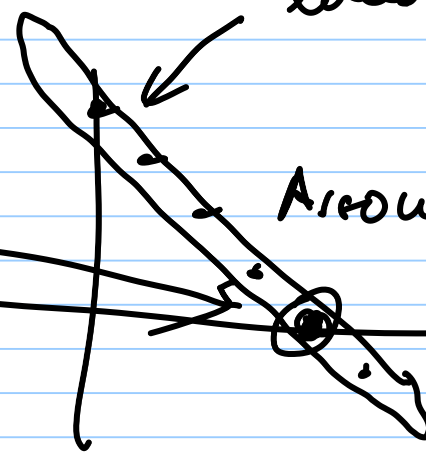
($\Leftarrow$ source or ~~target~~ always 0)

We say spectral sequence <u>collapses</u>

$$E^{\infty}_{pq} = \begin{cases} 0 & q \neq 0 \\ H_p^{cell}(X;A) & q = 0 \end{cases}$$

Associated graded
only one non-trivial
piece for each $n = p+q$

therefore



$$H_p(X;A) = H_p^{cell}(X;A).$$

This proves the cell homology theorem.

Example: "A trivial generalization"

Suppose I give for each $n \in \mathbb{Z}$ an abelian group A_n. I can define a generalized homology th

$$E_m(X) = \bigoplus_{n \in \mathbb{Z}} H_{m-n}(X; A_n).$$

$$\boxed{E_n(*) = A_n \qquad n \in \mathbb{Z}}$$

Homology commutes with $\oplus$

AHSS collapses for E_*.

We still call this an ordinary homology theory.

<u>Example</u>: K-theory homology.
(comes more naturally as cohomology)

Reduced form of AHSS:

$$\tilde{E}^2_{pq} = \tilde{H}_p(X; E_q(*)) \Rightarrow \tilde{E}_{p+q}(X).$$

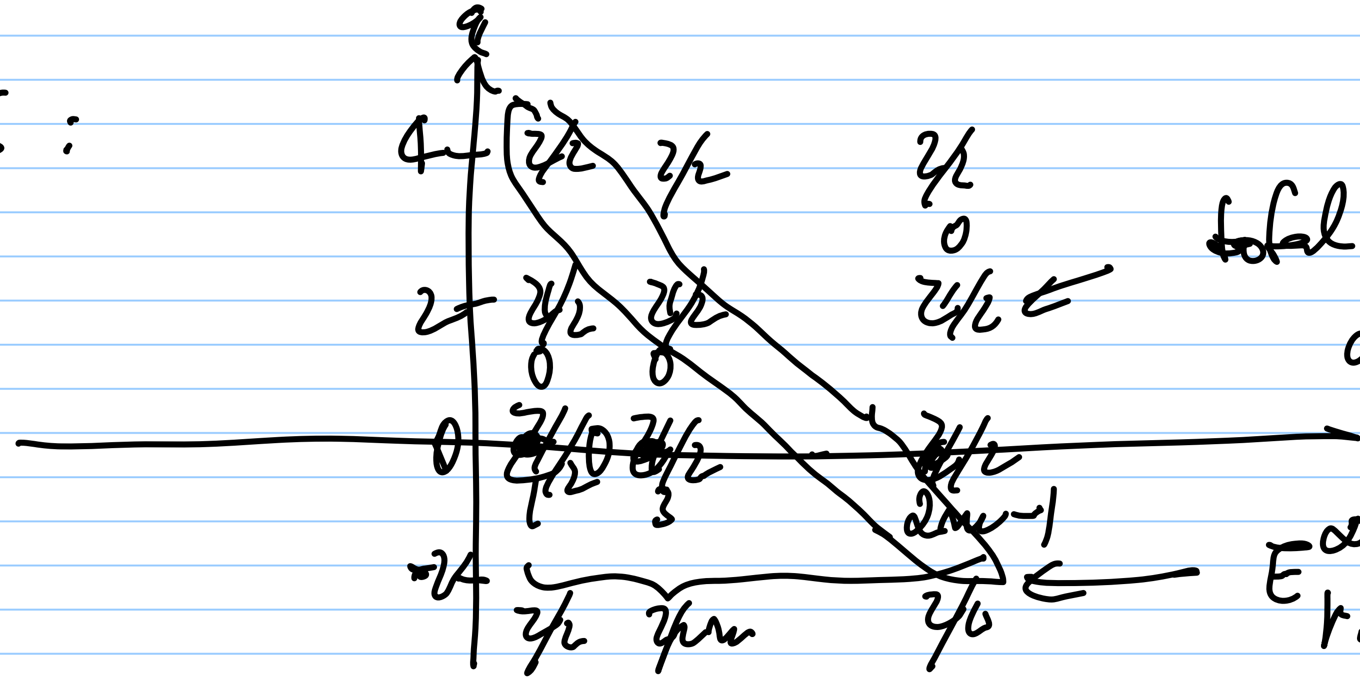
K-theory homology:

$$K_q(*) = \begin{cases} \mathbb{Z} & q \text{ even} \\ 0 & q \text{ odd} \end{cases}$$

$X = \mathbb{R}P^{2m}$

$$\tilde{H}_p(X) = \begin{cases} \mathbb{Z}/2 & 0 < q < 2n \text{ odd} \\ 0 & \text{else.} \end{cases}$$

AHSS :



total degree odd

$d^r = 0 \quad r > 2$

$E^{\infty}_{p,q}$ ← n copies of $\mathbb{Z}/2$

$p+q$ constant

Again, the spec. seq. collapses

In fact,

$$\tilde{K}_{2i+1}(\mathbb{RP}^{2m}) = \mathbb{Z}/(2^m).$$