Note Title 9/26/2022

MATH 695 9/26/2022

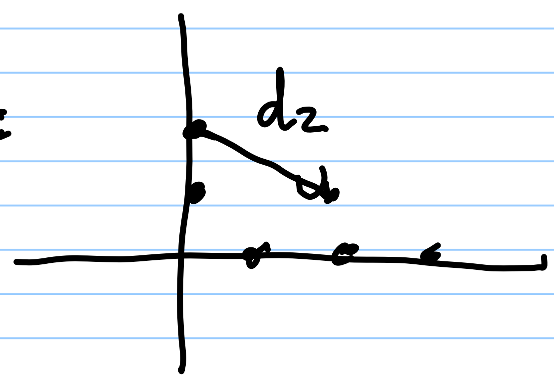
<u>Cohomological spectral sequences</u> E_r^{pq}

(like homological, where you change signs of p,q

$$E^r_{-p,-q} = E_r^{pq} \;)$$

$$d_r : E_r^{pq} \to E_r^{p+r,\, q-r+1}$$

<u>Example</u>:



E_2

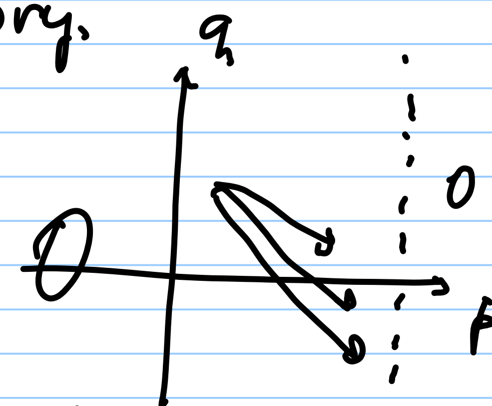
$$E_{r+1} = H(E_r; d_r)$$

Atiyah-Hirzebruch spectral sequence in cohomology

$$E_2^{pq} = H^p_{cell}(X; E^q(*)) \Rightarrow E^{p+q}(X)$$

E a generalised cohomology theory.

X finite-dimensional CW-complex.



(In general, true when it converges algebraically — conditional convergence.)

A little more homotopy theory (motivated by the question: how to construct a generalized cohomology theory?)

Based = The category of based topological spaces, continuous maps preserving the base point.

This category has a 0 : $*$

initial $\cong$ terminal object

Recall the based mapping cone $\tilde{C}f =$
$f: Y \to X$ based map

$$= X \amalg Y \times [0,1] \Big/ \begin{array}{l} (y,0) \sim f(y) \\ (y,1) \sim (y',1) \\ (*,t) \sim (*,t') \end{array}$$

<u>Lemma</u>: Let $f: Y \to X$ be a based map, let Z be a based space. Then we have an exact sequence of based sets

"i^*" "f^*" $\leftarrow$ $\mathrm{Ker}\, g = g^{-1}(0)$

$$[\tilde{C}f, Z] \xrightarrow{[i, Z]} [X, Z] \xrightarrow{[f, Z]} [Y, Z]$$

$i : X \to \tilde{C}f$ is the based inclusion.

$[Y, Z] = \{$ based homotopy classes of based maps $Y \to Z \}$.

based homotopy $h : Y \times [0,1] \to Z$

$h(y,t) = h_t(y)$ $\qquad h(*, t) = *$

$h_0(y) = h(y,0) = f(y)$ $\qquad f \simeq_{\text{based}} g.$

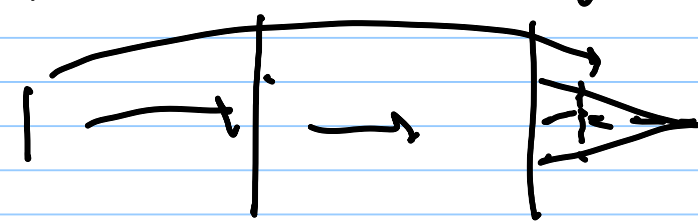
$h_1(y) = h(y,1) = g(y)$

Proof: $f^* i^* \overset{?}{=} 0$ $\qquad$ $\mathrm{Im}\, i^* \subseteq \mathrm{Ker}\, f^*$

$\quad\;\; \|$ $\qquad\qquad\qquad\qquad$ $if \simeq$

$\;\;(if)^*$

$$Y \xrightarrow{f} X \longrightarrow C_f$$



This is iff. Suppose that we have $\varphi : X \to Z$

such that $\varphi \circ f : Y \to Z$.

based $\simeq \to h_f \simeq_{\text{based}} 0$

$\Big\}\;\; \mathrm{Im}\, i^* \supseteq \mathrm{Ker}\, f^*.$

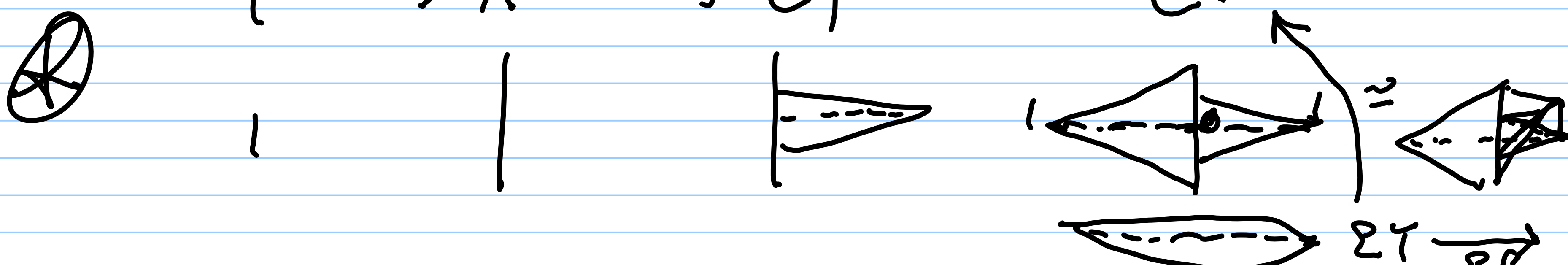
$h_0 = \varphi \circ f \qquad h_1 = 0$

Extend φ to $\tilde{\varphi} : \tilde{C}f \to Z$

$\tilde{\varphi}(x) = \varphi(x) \quad x \in X$

$\tilde{\varphi}(y,t) = h(y,t).$ $\square$

$$\begin{tikzcd}
Y \arrow[r, "f"] & X \arrow[r, "i"] & \tilde{C}f \arrow[r, "j"] & \tilde{C}i \arrow[r, "k"] & \tilde{C}j \\
 & & & \Sigma Y \arrow[u, "\simeq"] \arrow[r, "-\Sigma f"'] & \Sigma X \arrow[u, "\simeq"']
\end{tikzcd}$$



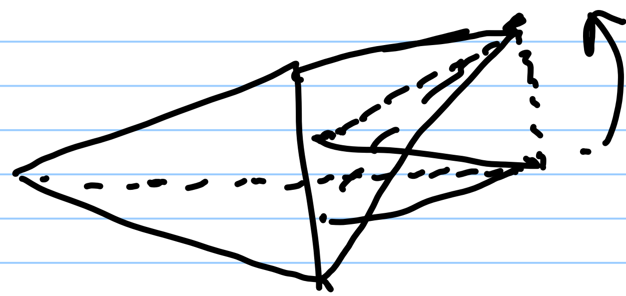
based-homotopy equivalent

$$\tilde{C}_i \underset{\text{based}}{\simeq} \Sigma Y = Y \times [0,1] \Big/ \begin{array}{l} (y,0)\sim(y',0) \\ (y,1)\sim(y',1) \\ (*,t)\sim(*,t') \end{array}$$

reduced ↑ reverse coordinates

$$\tilde{C}_i \longrightarrow \Sigma Y$$

$$\searrow \tilde{C}_i / \widetilde{CX} \;\text{''}\!\!\wr$$



collapse

HW ① Prove that $C_i \simeq \Sigma Y$, $C_j \simeq \Sigma X$.
(in coordinate).

<u>Up to based $\simeq$</u> ← preserves $[\,,\,]$,

⊛ means that if we keep taking based mapping cones, we get a sequence

$$Y \xrightarrow{f} X \xrightarrow{i} \tilde{C}f \longrightarrow \Sigma Y \xrightarrow{-\Sigma f} \Sigma X \longrightarrow \overbrace{\Sigma \tilde{C}f}^{\substack{\tilde{C}\Sigma f \\ \cong}} \longrightarrow \Sigma^2 Y \xrightarrow{\Sigma^2 f} \Sigma^2 X \to \cdots$$

So for a based space Z, we get a long exact sequence

$$\underbrace{[\Sigma C_f, Z] \to [\Sigma X, Z] \xrightarrow{-\Sigma f^*} [\Sigma Y, Z]}_{\text{groups}} \to [C_f, Z] \xrightarrow{i^*} [X, Z] \xrightarrow{f^*} [Y, Z]$$

$$\begin{array}{c} [\Sigma^2 Y, Z] \\ \uparrow \Sigma^2 f \\ [\Sigma^2 X, Z] \\ \uparrow \\ \vdots \end{array}$$

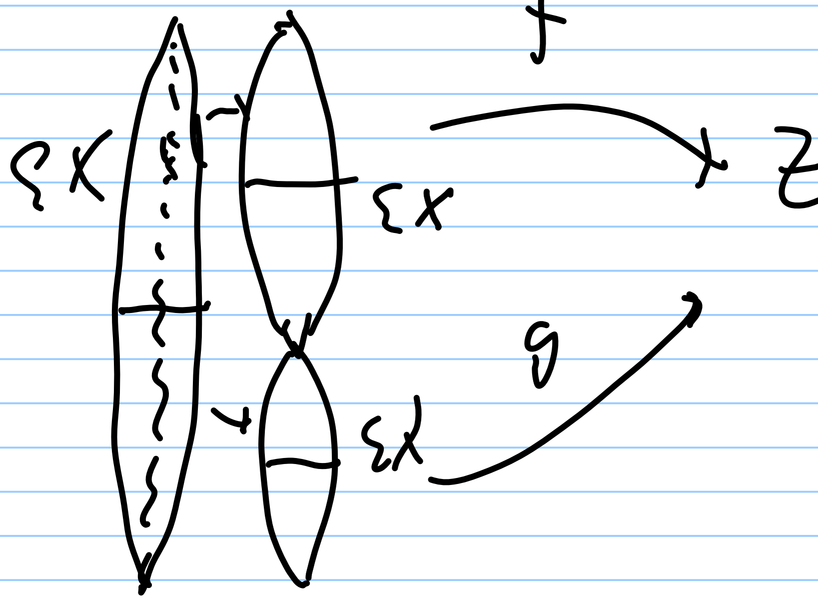
abelian groups

$[\Sigma X, Z]$ is a group

~~X any based spaces~~

$[\Sigma^2 X, Z]$ is an ab. group.

just a slight generalisation of the construction of $\pi_1 Z = [S^1, Z] = [\Sigma S^0, Z]$



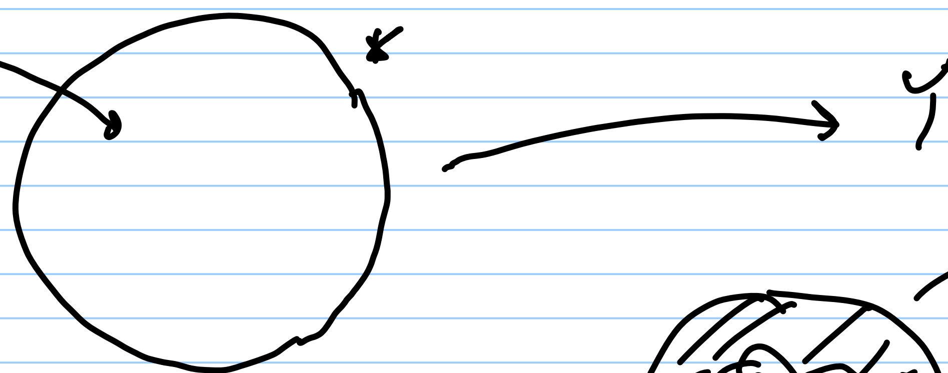
vice versa = reversing suspension coords.

$\Sigma^2 X \simeq \Sigma\Sigma X$ $\qquad [\Sigma^2 X, Y]$

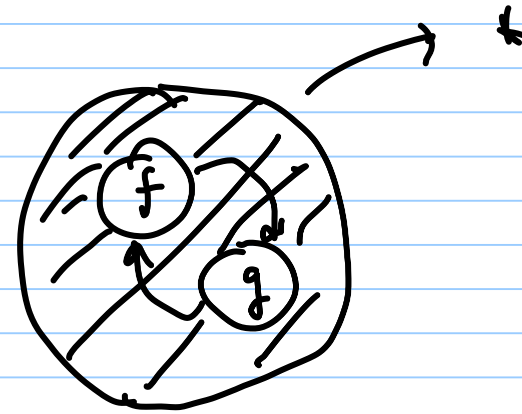
commutative

S. Mac Lane

decorated w/ pts of X



addition



important in alg. topology

For future reference: Z hand

nth homotopy group of Z

$$[\underbrace{\Sigma \cdots \Sigma}_{n} S^0, Z] = [S^n, Z] = \pi_n Z$$

abelian group for $n > 1$.

generalised cohomology theories