MATH 695 9/28/2022

Note Title 9/28/2022

$f : X \to Y$

$\Sigma f : \Sigma X \to \Sigma Y$

$(x,t) \mapsto (f(x), 1-t)$

$$\Sigma X = X \times [0,1] \Big/ \begin{matrix}(x,0)\sim(x',0) \\ (x,1)\sim(x',1) \\ (*,t)\sim(*,t')\end{matrix}$$

The smash-product of based spaces X, Y

$$X \wedge Y = X \times Y \big/ \{*\} \times Y \cup X \times \{*\}$$

F. Adams

$$S^m \wedge S^n \cong S^{m+n}$$

$$[0,1]^n/\partial[0,1]^n \times [0,1]^m/\partial[0,1]^m = [0,1]^{n+m}/\partial[0,1]^{n+m}$$

$$\Sigma X \cong X \wedge S^1$$

If K compact Hausdorff, in Hausdorff spaces,

$K \times ?$ has a right adjoint $\mathrm{Cont}(K, ?)$

compact-open topology

(Fails in general. X <u>compactly generated</u> if for a subset $Z \subseteq X$, Z is closed iff $\forall K \subseteq X$ compact $K \cap Z$ is closed in K.

continuous maps

convergent seq + limit

Metric spaces, Manifolds, CW-complexes are compactly generated

X comp. gen. <u>Weakly Hausdorff</u>: $X \to X \times X$ is closed

compactly generated product

$\boxed{\text{May: Concise course}}$

"Bait-and-switch": we work in the category of compactly generated weakly Hausdorff spaces

In this category, the (curr) exponential adjunction holds in general. (Steenrod MMJ)

e.g. in the category of compactly generated weakly Hausdorff spaces, a product of two CW-complexes is a CW-complex

(false in Top - Milnor)

"Bait-and-switch:" Top from now on means compactly generated weakly Hausdorff spaces

We also have an adjunction: Z any space
$\underbrace{\text{an adjunction}}_{\text{based exponential}}$

in Based
$$\begin{cases} Z \wedge ? & \text{has right adjoint } F(Z, ?) \quad \leftarrow \text{subspace of based maps} \\ \Sigma ? = ? \wedge S^1 & \text{has right adjoint } F(S^1, ?) = \Omega ? \quad \leftarrow \text{space of based loops} \end{cases}$$

Passes to based homotopy classes.

Recall from last time that for $f : Y \to X$ based, Z based space, we have a long exact sequence

based sets

groups

based homotopy classes

$$[\Sigma \tilde{C}f, Z] \to [\Sigma X, Z] \xrightarrow{-f^*} [\Sigma Y, Z] \to [\tilde{C}f, Z] \to [X, Z] \xrightarrow{f^*} [Y, Z]$$

$$\nwarrow [\Sigma^2 Y, Z] \xleftarrow{\Sigma^2 f^*} [\Sigma^2 X, Z] \leftarrow \cdots$$

abelian groups.

$\circledast$ Now assume I have a sequence of based spaces $(Z_n)_{n\in\mathbb{Z}}$ together with homotopy equivalences:

$$Z_n \xrightarrow{\simeq} \Omega Z_{n+1} \qquad n \in \mathbb{Z}$$

<u>Theorem</u>: The data $\circledast$ give rise to a generalised cohomology theory on <u>based CW-cxes</u>:

$$\tilde{E}^n(X) := [X, Z_n].$$

The converse is <u>also true</u>. Any based generalized

cohomology theory $\tilde{E}$ on based CW-cxes (which satisfies
$$\tilde{E}^n\Big(\bigvee_{i\in I} X_i\Big) \xrightarrow{\cong} \prod_{i\in I} \tilde{E}^n X_i \quad)$$
arises from the data ⊛.

Proof ⇒ Given $(Z_n)_{n\in\mathbb{Z}}$, the suspension axiom:

$$\tilde{E}^{n+1}\Sigma X = [\Sigma X, Z_{n+1}] \overset{\text{adjunction}}{\cong} [X, \Omega Z_{n+1}]$$

$$\qquad\qquad \| \wr \text{ our data}$$

$$\tilde{E}^n X \cong [X, Z_n]$$

automatically, $\tilde{E}^{n+2}\Sigma^2 X = [\Sigma^2 X, Z_{n+2}]$ is an abelian group. So we have suspension and exactness. ($\vee$ axiom $\Leftarrow$ adjunction)

<u>Converse</u>: Brown representability Theorem. $\square$

<u>Example</u>: $U(n)$ = group of $n \times n$ complex matrices A such that $A\bar{A}^T = I$

$$U(n) \subset GL(n, \mathbb{C}) \leftarrow \text{all complex invertible } n\times n \text{ matrices}$$

$\uparrow$ Gram-Schmidt $\simeq$

$$U(n) \subset U(n+1)$$
$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

$$GL(n, \mathbb{C}) \subset GL(n+1, \mathbb{C})$$

$$U = \bigcup U(n)$$

$$GL(\mathbb{C}) = \bigcup GL(n, \mathbb{C}) \simeq U$$

<u>Theorem</u> (Bott): $\Omega^2 U \simeq U$. (Bott periodicity theorem).

So we know this gives rise to a generalised cohomology theory $\Leftarrow$ K-theory (complex)

<u>Convention</u>:

$$Z_{2n+1} := U \qquad n \in \mathbb{Z}$$

$$Z_{2n} := \Omega U$$

Define $\tilde{K}^n X := [X, Z_n]$.

(Recall $K^n X = \tilde{K}^n X_+$ ← disjoint base point.)

<u>Fact</u>: $\pi_1 U(n) = \mathbb{Z}$

$U(n) \subset U(n+1)$

$\pi_1 U(n) \xrightarrow{\cong} \pi_1 U(n+1)$

$\pi_1 U = \mathbb{Z}$

$\therefore K^{2n}(*) = \mathbb{Z}$

$K^{2n+1}(*) = 0$

$K^{2n}(*) = \tilde{K}^{2n}(S^0) = [S^0, \Omega U] =$

$$= [S^1, U] = \pi_1(U) = \mathbb{Z}$$

(unbased = classes of maps $* \to U$)

$$K^{2n+1}(*) = \tilde{K}^{2n+1}(S^0) = [S^0, U]$$

$$= \pi_0 U = 0$$

path-connected.

HW (2) Use the cohomological Atiyah-Hirzebruch spectral sequence to calculate $K^{n}(\mathbb{C}P^n)$.

Comment: ⊛-data specifying a generalised cohomology theory: $(Z_n)_{n\in\mathbb{Z}}$ based spaces

$$Z_n \xrightarrow{\simeq} \Omega Z_{n+1}.$$ delooping

The spaces Z_n are called <u>infinite loop spaces</u>.

It suffices to have Z_n defined for $n \geq 0$. ($n \geq N$)

$$Z_{-1} := \Omega Z_0,\ Z_{-2} := \Omega^2 Z_{-1} \cdots .$$