MATH 695 9/30/2022

Note Title 9/30/2022

May: Concise Course
Chapters 6-10

Fibrations

dual to cofibrations.

A map $f: X \to Y$

is a fibration if it satisfies the Homotopy lifting property: Given a "testing map" $g: Z \to X$ and a homotopy $h: Z \times [0,1] \to Y$ with $h_0 = f \circ g$. Then there exists a homotopy

$H : Z \times [0,1] \to X$ such that $H_0 = g$ and $f \circ H_t = h_t$.

$$\begin{tikzcd} z \arrow[d, mapsto] & Z \arrow[r, "g"] \arrow[d] & X \arrow[d, "f"] \\ (z,0) & Z \times [0,1] \arrow[r, "h"] \arrow[ur, dashed, "H"] & Y \end{tikzcd}$$

<u>Note</u>: Same as in covering space theory except uniqueness. In particular, a covering is a fibration. (in compactly generated setup)

<u>Theorem</u>: If $f: X \to Y$ is a map which is a fiber bundle which means that for every $y \in Y$ $\exists$ U open $y \in U$ such that we have a diagram

$$\begin{tikzcd} F \times U \arrow[r, "\simeq"] \arrow[d, "\mathrm{proj}"'] & f^{-1}(U) \arrow[d] \\ U \arrow[r, equal] & U \end{tikzcd}$$

(note: $\underset{\leftarrow \text{fiber at } y}{F}$ can depend on y)

("locally a product") and <u>Y is paracompact</u> then f is a fibration.

Note: A covering is a fiber bundle with discrete fibers.

Y Paracompact: Every open cover (U_i) of Y has a locally finite refinement (V_j).

refinement means $\forall j\ \exists i\ V_j \subseteq U_i$

locally finite means $\forall y \in Y\ \exists U$ open $y \in U$ $\quad U \cap U_i \neq \emptyset$ only for finitely many i.

CW - complexes are paracompact.

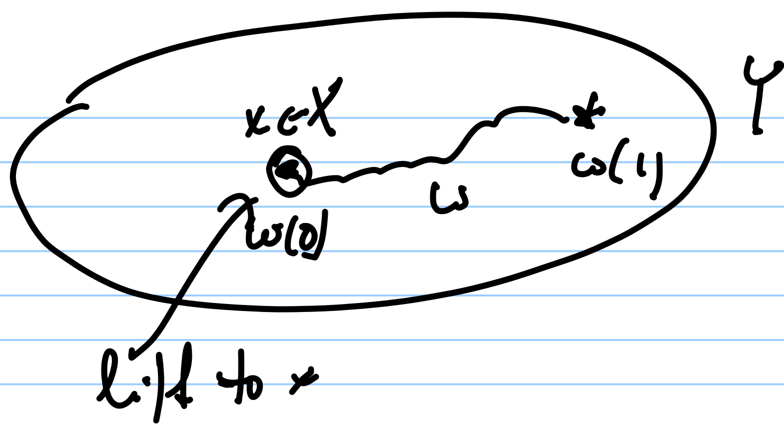
$\cup\!|$

topological manifold

Now suppose $f : X \to Y$ is a based map.

The homotopy fiber Ff as follows:

$$Ff = \left\{ (x, \omega) \in X \times \underbrace{\mathrm{Cont}([0,1], Y)}_{\text{space of paths}} \;\middle|\; \begin{array}{l} \omega(0) = f(x) \\ \omega(1) = * \end{array} \right\}$$



Dual to the construction to the mapping cone

$$F(f) = \lim \left(\begin{tikzcd} X \arrow[r, "f"] & Y \\ & \mathrm{Cont}([0,1], Y) \arrow[u, "ev_0"] \arrow[d, "ev_1"] \\ * \arrow[r] & Y \end{tikzcd} \right)$$

If you turn around all the arrows replace lim with colim and $\mathrm{Cone}([0,1],Y) = F([0,1]_+, Y)$ ← based maps

by $[0,1]_+ \wedge Y$

you get $\tilde{C}f$ ← based mapping cone.

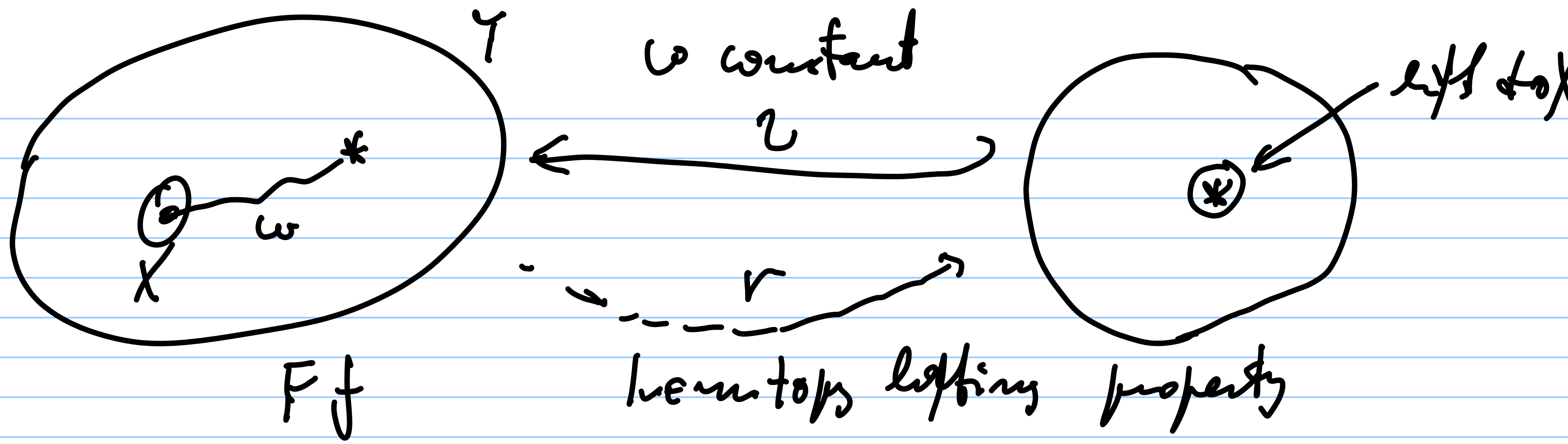
<u>Proposition</u>: If $f: X \to Y$ is a fibration then

$Ff \simeq f^{-1}(*)$

homotopy equivalent

actual fiber

<u>Proof</u> dual to the proof that $Cf \simeq X/Y$ if $f: Y \to X$ is a cofibration



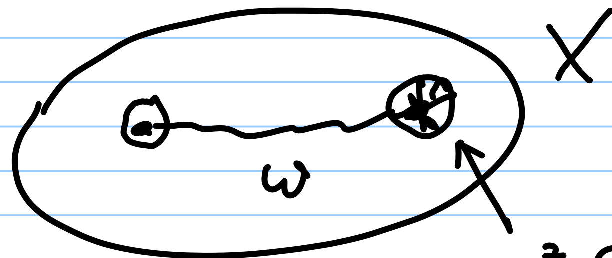
$$h: Ff \times [0,1] \longrightarrow Y$$
$$((x,\omega),t) \longmapsto \omega(t)$$

lift it to

$$H: Ff \times [0,1] \longrightarrow X$$

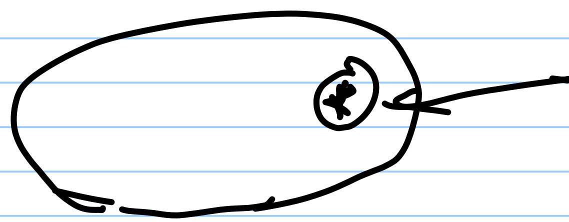
take H_1.

Both homotopies come from the map H:



$z \in f^{-1}(*)$

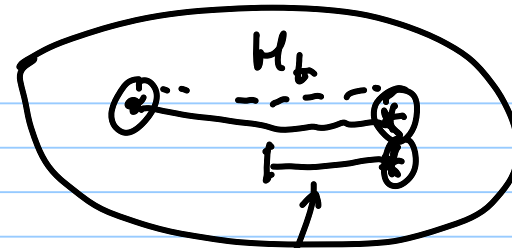
$ri \simeq Id_{f^{-1}(*)}$



lifting along the constant path tells us H_1

homotopy: $H_t(x) \in f^{-1}(*)$

$$ir \simeq \mathrm{Id}_{F(f)}$$

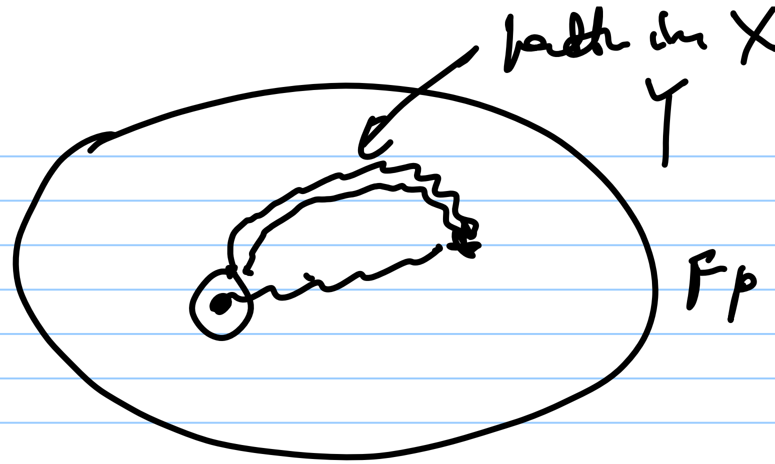


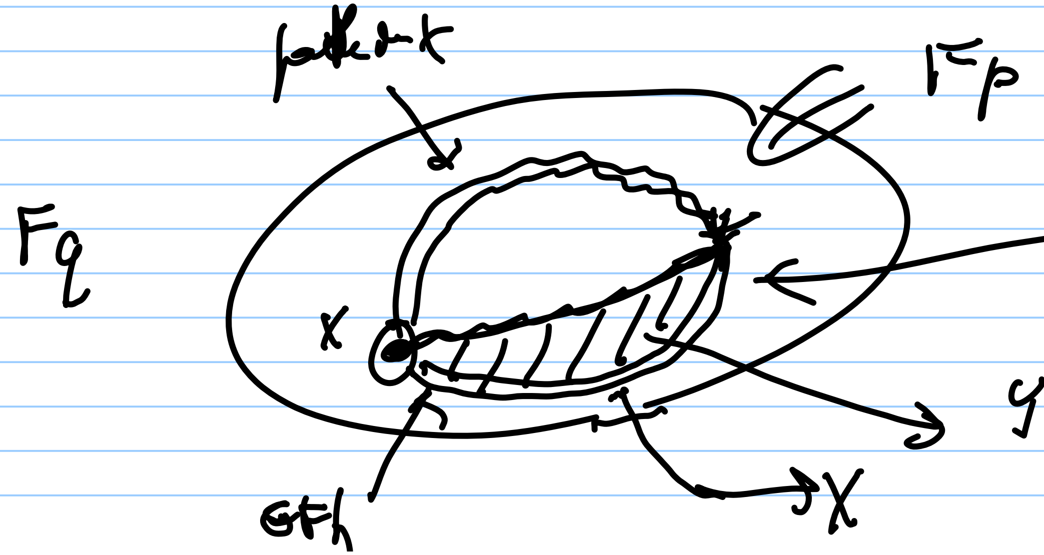
at time $t \in [0,1]$
reparametrize to
$[t,1]$

$\square$

$$\begin{tikzcd}
Fq \arrow[r] & Fp \arrow[r, "q"] & Ff \arrow[r, "p"] & X \arrow[r, "f"] & Y \\
\Omega X \arrow[u, phantom, "\wr"] \arrow[r, "\Omega f"'] & \Omega Y \arrow[u, phantom, "\wr"] & (x,\omega) \arrow[r, mapsto] & x &
\end{tikzcd}$$

$\Omega X \xrightarrow[\Omega f]{} \Omega Y$ ∈ loops l commutes up to homotopy





bottom half: path in F_f

<u>Lemma</u>: Let Z be a based space and $f: X \to Y$ a based map. Then we have an exact sequence of based sets

$$[Z, Ff] \xrightarrow{[Z,p]} [Z, X] \xrightarrow{[Z,f]} [Z, Y].$$

HW 3 (dual to the proof for cofibration)

Theorem: In the situation of the lemma, we have a long exact sequence

$$
\begin{tikzcd}
{[Z,Ff]} \arrow[r] & {[Z,X]} \arrow[r, "f_*"] & {[Z,Y]} \\
{[Z,\Omega Ff]} \arrow[r] & {[Z,\Omega X]} \arrow[r, "-\Omega f_*"] & {[Z,\Omega Y]} \arrow[ull] \\
\cdots \arrow[r] & {[Z,\Omega^2 X]} \arrow[r, "\Omega^2 f_*"] & {[Z,\Omega^2 Y]} \arrow[ull]
\end{tikzcd}
$$

LES of abelian groups.

of groups

$\square$

Special cases of motc: $Z = S^0$. LES of homotopy groups

$$\begin{tikzcd}
\pi_0 Ff \arrow[r] & \pi_0 X \arrow[r] & \pi_0 Y \\
\pi_1 Ff \arrow[r] & \pi_1 X \arrow[r] & \pi_1 Y \arrow[ull] \\
\pi_2 Ff \arrow[r] & \pi_2 X \arrow[r] & \pi_2 Y \arrow[ull]
\end{tikzcd}$$

LES of groups

ab. groups

Two special cases: If f is a fibration, you can replace Ff by $F = f^{-1}(*)$ (some applications next time)

If $f = i : X \hookrightarrow Y$ is an inclusion,

then define $\pi_n(Y, X) := \pi_{n-1}(F(i))$

relative homotopy group

group for $n \geq 2$
abelian for $n \geq 3$

Then we have an (imperfect) analog of LES in homology

fails in low dim.

(first fails to be abelian
then fails to be a group
then fails to extend)

Homotopy groups lack excision.

(however, it holds in a dimensional range
$\Rightarrow$ stable homotopy theory

is a generalized (homology)