MATH 695 10/3/2022

Note Title 10/3/2022

Recall from last time: If we have a sequence of based spaces

$$F \longrightarrow E \xrightarrow{f} B$$

where f is a fibration and $F = f^{-1}(*)$ then we have a long exact sequence

$$\pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \cdots \to \pi_0(F) \to \pi_0(E) \to \pi_0(B)$$

Example 1: If f is a based covering: F discrete

$\pi_i F = 0 \quad i > 0$

$$\underbrace{0 \to \pi_n(E) \to \pi_n(B) \to 0}_{\boxed{\pi_n(E) \cong \pi_n(B) \text{ for } n \geq 2}} \qquad n > 1$$

In particular, if the universal cover of X ($= B$) is contractible (X path-connected l.c., SLSC) then $\pi_n(X) = 0$ for $n \geq 2$. If $\pi = \pi_1(X)$

we say X is of type $K(\pi, 1)$.

Examples: Any (connected) graph $K(F_n, 1)$ ← free group

$$S^1 = K(\mathbb{Z}, 1)$$

Any (comm) surface except $S^2, \mathbb{RP}^2$ is $K(\pi, 1)$

Any hyperbolic manifold (has a Riemann metric with scalar curvature < 0).

Example: If G is a say, compact Lie group acting freely on a manifold M then

$$G \to M \to M/G$$

is a fiber bundle, therefore a fibration,

$$\cdots \to \pi_n G \to \pi_n M \to \pi_n M/G \to \pi_{n-1}(G) \to$$

S^1 acts on $S^{2n+1} = \{(z_0, \ldots, z_n) \in \mathbb{C}^n \mid |z_k|^2 = 1\}$

$\shortparallel$

$\{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ $\qquad \lambda(z_0, \ldots z_n) = (\lambda z_0, \ldots \lambda z_n)$

$$S^{2n+1}/S^1 = \mathbb{C}P^n$$

Fibration : $S^1 \to S^{2n+1} \to \mathbb{C}P^n$

Therefore, for $k \geq 3$

$$\underbrace{\pi_k S^1}_{0} \to \pi_k S^{2n+1} \to \pi_k \mathbb{C}P^n \to \underbrace{\pi_{k-1} S^1}_{0}$$

$$\boxed{\text{For } k \geq 3 \quad \underset{n \geq 1}{\pi_k \mathbb{C}P^n} \cong \pi_k S^{2n+1}}$$

$n = 1$ case : $\mathbb{C}P^1 \cong S^2$

$\therefore \quad \pi_k(S^2) \cong \pi_k(S^3) \quad$ for $k \geq 3$

But $\pi_3 S^3 = \mathbb{Z}$

$$\therefore \quad \boxed{\pi_3 S^2 \cong \mathbb{Z}} \qquad \text{(Hopf)}$$

Compose Hopf-map with degree k map $S^3 \to S^3$

$k \in \mathbb{Z} \cong \pi_3 S^2$

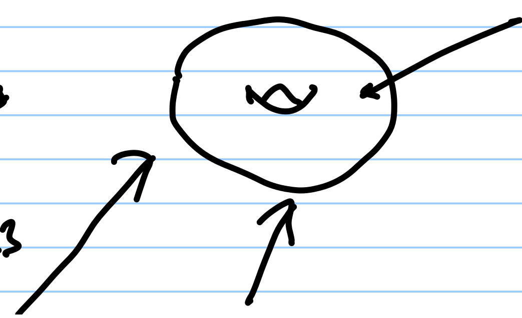
Hopf fibration

$$S^3 \longrightarrow S^2$$

torus

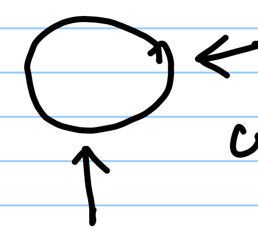
$S^1 \times D^2$

$\cup\ D^2 \times S^1 = S^3$



inside: solid torus

in the middle of solid torus



radial circle

gluing another solid torus outside

wire on torus

spin a wire around it

in 360°, rotate once around radial circle

glue two foliations together

$\Rightarrow$ Hopf fibration

$\pi_k(S^2)$

1	0
2	$\mathbb{Z}$
3	$\mathbb{Z}$
4	?
5	?

$\pi_4 S^3 = \mathbb{Z}/2$

Stable homotopy groups

$\pi_n^s(X) = \operatorname{colim}_k \pi_{n+k} \Sigma^k X$

based generalized homology theory = stable homotopy groups

X f.d. $\Rightarrow$ eventually constant

n	$\pi_n^s(S^0)$
0	$\mathbb{Z}$
1	$\mathbb{Z}/2$
2	$\mathbb{Z}/2$
3	$\mathbb{Z}/24$
4	0
5	0
	$\vdots$

up to dim. ~ 14 all have form

geometric interpretation (MIT undergraduate topology project)

$E(8) \sim K(\mathbb{Z}, 3)$

exceptional Lie group

up to these dimensions ~14

Let us return to G compact Lie acting freely on a CW-complex.

$$G \to X \xrightarrow{f} X/G \quad \} \text{ a fibration}$$

We want to think about the case when

X is contractible : $X \simeq *$. Then the homotopy fiber

$$Ff \simeq \Omega X/G = F(* \to X/G)$$

preserves based homotopy equivalences

$$\wr\!\wr$$

$$G.$$

$$\boxed{G \simeq \Omega X/G}$$

Examples: $\mathbb{Z}/2$ acts on $S^\infty = \bigcup S^n$ unit sphere in $\mathbb{R}^{n+1}$

$$x \longmapsto -x$$

Quotient: $\mathbb{RP}^\infty = \bigcup \mathbb{RP}^n$

$$\boxed{\Omega \mathbb{RP}^\infty \simeq \mathbb{Z}/2}$$

in particular, $\mathbb{RP}^\infty$ is of type $K(\mathbb{Z}/2, 1)$.

In fact, $\mathbb{Z}/k \subset S^1$ acts freely on $S^\infty = \bigcup S^{2n+1}$

unit sphere in $\mathbb{C}^{n+1}$

$$S^\infty / \mathbb{Z}/k = L^\infty(k)$$

infinite lens space

of type $K(\mathbb{Z}/k, 1)$

$$\mathbb{Z}/k \simeq \Omega L^{\infty}(k)$$

S^1 acts on the unit sphere in $\mathbb{C}^{\infty}$: Quotient

$$\mathbb{C}P^{\infty} = \bigcup_n \mathbb{C}P^n \qquad \boxed{\Omega \mathbb{C}P^{\infty} = S^1}$$

$K(\mathbb{Z},1)$

$\mathbb{C}P^{\infty}$ is of type $K(\mathbb{Z}, 2)$

X of Type $K(G, n)$: $\pi_n(X) = G$

path-connected

$\pi_i(X) = 0 \quad i \neq n$

(G abelian if $n > 1$).

Eilenberg - Mac Lane spaces.

$\Omega K(G, n) = K(G, n-1)$

$E_n = K(G, n)$ $\quad n \geq 1 \leftarrow$ Ω to get $E_{<0} \simeq *$

We get a generalized cohomology theory

on CW-complexes:

$$\tilde{E}^n X = [X, K(G,n)] \qquad \boxed{E_0 = G}$$

based

What is it? It satisfies the dimension axiom!

$$\therefore \quad \boxed{\tilde{E}^n X = H^n(X;G)}$$

HW: 1. Prove that if $f : X \to Y$ is

a fibration and Y is path-connected, then for $x, y \in Y$, $f^{-1}(x) \simeq f^{-1}(y)$.

② If we have a diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \\ X' \arrow[ur, "f'"'] & \end{tikzcd}$$

where g is a homotopy equivalence, prove that $F(f) \simeq F(f')$.