Bott periodicity theorem: $\Omega^2 U \simeq U$

G ← compact Lie acts freely on a CW-complex E which is contractible $E \simeq *$ then we have a fibration

$$G \to E \to E/G$$

where moreover $G \simeq \Omega(E/G)$ $\left(= F(* \to E/G) \right)$

We then write $BG = E/G$.

In the case of $G = U(n)$, we have the space of m-tuples of orthonormal vectors in $\mathbb{C}^{m+n}$. This is the Stiefel manifold $St(m,n)$. $U(n)$ acts freely on $St(m,n)$ so we have a fibration

$$U(n) \longrightarrow St(m,n) \longrightarrow \underbrace{St(m,n)/U(n)}_{Gr(m,n)}$$

(of course, $St(m,n) \not\simeq *$).

complex Grassmannian

$\parallel$

space of n-dimensional

but $\bigcup_{m} St(m,n) \simeq *$.   complex vector subspaces of $\mathbb{C}^{m+n}$)

Milnor trick:

$\mathbb{C}^{\infty} = \bigoplus_{\infty} \mathbb{C}$ (convention)

$h_t : \mathbb{C}^{\infty} \longrightarrow \mathbb{C}^{\infty}$

$t(x_1, x_2, \dots \quad) + (1-t)(x_1, 0, x_2, 0, \dots) \neq 0$

eventually 0

Look how it transforms a frame (Gram–Schmidt)

Second linear homotopy $\mathbb{C}^\infty \to \mathbb{C}^\infty$

$$h_t(x_1, x_2, \dots) = t(x_1, 0, x_2, 0, \dots)$$
$$+ (1-t)(0, 1, 0, 0, \dots)$$

Apply to frames, Gram-Schmidt $\Rightarrow$ $St(\infty, n) = U(\mathbb{C}(\infty, n))$

$\simeq *$

Taking a colimit, we have a fibration

$$U(n) \to St(\infty, n) \overset{\simeq *}{\to} St(\infty, n)/U(n)$$

$$\underset{\sim}{U} Gr(\infty, n) = BU(n)$$

$$\Omega BU(n) \simeq U(n)$$

$\aleph_0 + \aleph_0 = \aleph_0$ add more coordinates and take union over n

$$\bigcup_n BU(n) =: BU$$

$$\boxed{\Omega BU \simeq U}$$

To prove Bott periodicity, it suffices to prove that

$$\boxed{\Omega U \simeq BU \times \mathbb{Z}}$$

$$\coprod_{n \in \mathbb{Z}} BU$$

select a base point @ $n=0$

$$\Omega(BU \times \mathbb{Z}) \simeq \Omega BU \simeq U$$

$$\pi_0 \Omega X = [S^0, \Omega X] = [S^1, X] = \pi_1(X)$$

set of path-components

So we first need to prove $\pi_1(U) = \mathbb{Z}$.

HW2: Prove that $\pi_1 U(n) = \mathbb{Z}$. [Use the fibration

$$U(n-1) \to U(n) \longrightarrow S^{2n-1}.\ ]$$

act on first $n-1$ coord.

---

generator of $\pi_1 S^1$ is multiplication by $z \in S^1$

$\parallel$

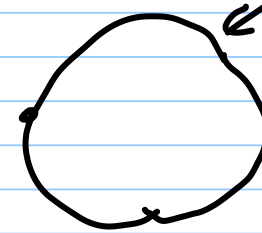
$\pi_1 U$

Analysis to prove Bott periodicity

Based loops in $U(n)$ $\quad (\in \Omega U(n))$

Can approximate them by Laurent loops.

$n \times n$ matrix



S^1

$p(z) =$

$= a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$

$(n \times n)$ matrices

some invertible

WOLOG polynomial multiplying by z

shifts us in the connected components of $\Omega(U(n))$

Homotopy method makes this function linear!

$$p_0 = p$$

$$z\, p_{r+1} = p_r(z) - p_r(0)$$

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_n \\ -z & 1 & & & \\ & -z & 1 & & \\ & & \ddots & \ddots & \\ & & & -z & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & p_1 & \cdots & p_n \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \underbrace{\begin{pmatrix} p & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}} \begin{pmatrix} 1 & & & \\ -z & 1 & & \\ & -z & \ddots & \\ & & -z & 1 \end{pmatrix}$$

check

adding a unit matrix
is crucial! (increasing n)

So, at this cost, I can assume

$F(z) \in SU(n)$ ← possibly increased

$\|$

$Az + B$ ← invertible for $z \in S^1$.

↑ A, B $n \times n$ matrices,

"Operator theory"

We can do it when $A, B \in \mathbb{C}$

$$\frac{1}{2\pi i}\int_{S^1}(Az+B)^{-1}\,d(Az+B)$$

1 if the 0 is inside S^1 $|z|<1$

0 otherwise.

$Az+B$ preserves

$Az+B$ invertible on $z \in S^1$

A, B $n\times n$ matrices $\in \mathrm{End}(V)$

f.d. complex vector space

This data determines a decomposition $V \cong V_0 \oplus V_\infty$ where on V_0, $Az+B$ is invertible for $|z|\leq 1$

V_0, V_∞

$t \in \mathbb{C}$

and on V_∞, $Az+B$

is invertible for $|z| \geq 1$

$\cup \{\infty\}$

$p(z) = Az + B$

$$Q_0 = \frac{1}{2\pi i} \int_{S^1} p(z)^{-1} \, d\, p(z) \qquad Q_\infty = \frac{1}{2\pi i} \int_{S^1} p(z) \, d\, p(z)^{-1}.$$

Q_0, Q_∞ are projection operators onto V_0, V_∞.

The data $(V_0, V_\infty) \in BU$.

This defines a homotopy equivalence between

$\Omega_0 U$ and BU

based loops homotopic to $*$.

Atiyah : K-theory.