MATH 695 10/7/2022

<u>Comment</u>: Using K-theory to construct elements in stable homotopy groups of spheres:

$$\begin{tikzcd} \alpha \in \pi_{2m+1}(U) \arrow[d, dotted] & \\ \pi_{2m+1}(U(N)) \arrow[u] & S^{2N} \to S^{2N} \arrow[l] \end{tikzcd}$$

1-pt. compactification of $\mathbb{C}^N$

based

maps of degree 1, can get it to degree 0 by adding in $\pi_{2N}S^{2N}$

α still not based, but can move base point to the base point :

finite group

$$\alpha \overset{\varepsilon}{\longmapsto} \pi^{s}_{2n+1}(S^{0})$$

(complex) J-homomorphism. Gives the best one can do at primes $p > 2$. At $p = 2$, use KO instead of K

repeat our construction for infinite orthogonal group O.

Theorem (Bott): $\Omega^8 O \simeq O$

real spinors

$\Omega O \simeq O/U$

$\Omega O/U \simeq U/Sp$

$\Omega U/Sp \simeq BSp \times \mathbb{Z}$

$\Omega BSp \times \mathbb{Z} \simeq Sp$

$USp(n)$

symplectic group (compact form)

like unitary group for quaternions.

$\Omega Sp \simeq Sp/U$

$\Omega Sp/U \simeq U/O$

$\Omega U/O \simeq BO \times \mathbb{Z}$

Atiyah: K-theory and reality

Take the proof of Bott periodicity we did and add complex conjugation. (All the steps we did preserve it!)

$\downarrow$

deduce real Bott periodicity via considerations with real spinors.

So we get finer information from $J : \pi_{2n+1} KO \to \pi^s_{2n+1} S^0$

$i \bmod 8$	$\pi_i KO$
0	$\mathbb{Z}$
1	$\mathbb{Z}/2$
2	$\mathbb{Z}/2$
3	0
4	$\mathbb{Z}$
5	0
6	0
7	0

Adams: On the groups $J(X)$ I - IV.

cyclic group of orders equal to denominators of Bernoulli numbers.

Back to a more general discussion.

What does algebraic topology see?

singular

homotopy equivalence of general spaces is ~~too~~ fine a relation. (homology, homotopy groups will never tell the Cantor set from the discrete set)

<u>Smaller goal</u>: The set of connected components??

What we can find is $\pi_0 X = \{\text{path - components}\}$

$[0,1] \to X$

Definition : A (continuous) map $f : X \to Y$ is called a weak equivalence if

$$\pi_0 f : \pi_0 X \xrightarrow{\cong} \pi_0 Y$$ bijection

and for all $n \geq 1$ and all $x \in X$

$$\pi_n f : \pi_n(X, x) \xrightarrow{\cong} \pi_n(Y, f(x)).$$

Algebraic topology works in a category where weak equivalences are made into isomorphisms (= derived category of spaces)

Also useful: "formal skeleton"

A map $f : X \to Y$ is called a <u>0-equivalence</u> if

$$\pi_0 f : \pi_0 X \to \pi_0 Y \text{ is } \underline{\text{onto}}$$

and an n-equivalence for $n > 0$ if

$$\pi_0 f : \pi_0 X \xrightarrow{\cong} \pi_0 Y$$

and for $k \geq 1$, $x \in X$

$$\pi_k f : \pi_k(X, x) \to \pi_k(Y, f(x))$$

is an isomorphism for $k < n$ and <u>onto</u> for $k = n$.

We have a category $\mathscr{C}$ with some class of morphisms $\mathcal{E}$ (usually we assume $\mathcal{E}$ forms a subcategory, includes all $\cong$ and satisfies the 2/3 property: $\xrightarrow{f}\xrightarrow{g}$, $\xrightarrow{h}$, $h = g \circ f$ if two out of the maps f, g, h are in $\mathcal{E}$, so is the third).

A derived category (if one exists) is a category $D\mathscr{C}$ together with a functor $\Phi : \mathscr{C} \to D\mathscr{C}$ which sends $e \in \mathcal{E}$ into an isomorphism and is universal with that property:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\Phi"] \arrow[dr, "F"'] & D\mathcal{C} \arrow[d, dashed, "D\Phi"] \\
& Q
\end{tikzcd}$$

F sends morphisms in $\mathcal{E}$ into isomorphisms then

$\exists!$ $D\Phi$ which makes the diag. strictly commute.

<u>Note</u>: Φ must be a bijection on objects.

Alternative 2-categorical definition which is invariant to equivalence of categories.

Why might a derived category not exist? (set theory)

HW 4 Prove that $U(1) \hookrightarrow U(n)$ is a 1-equivalence.