MATH 69.5 10/10/2022

Let $\mathcal{C}$ be a category $\mathcal{E} \subseteq \mathrm{Mor}\,\mathcal{C}$ a class of equivalences (subcategory, includes $\cong$, satisfies 2/3) then a derived category (equivalence-of-categories-proof definition) is a category $D\mathcal{C}$ together with a functor $\Phi : \mathcal{C} \to D\mathcal{C}$ which takes $\mathcal{E}$ into $\cong$ and additionally: For every functor $F : \mathcal{C} \to \mathcal{D}$ which sends $\mathcal{E}$ into $\cong$ there exists a functor $DF : D\mathcal{C} \to \mathcal{D}$ and a natural isomorphism

$\eta : F \longrightarrow DF \circ \Phi$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\Phi"] \arrow[dr, "F"'] & D\mathcal{C} \arrow[d, dotted, "DF"] \arrow[dl, phantom, "\Nearrow \eta"] \\
& \mathcal{D}
\end{tikzcd}$$

which is unique in the sense that for another such $\eta' : F \to D'F \circ \Phi$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\Phi"] \arrow[dr, "F"'] \arrow[dr, phantom, shift left=3, "\cong\eta"] & D\mathcal{C} \arrow[d, bend right=40, "DF"'] \arrow[d, bend left=40, dashed, "D'F"] \arrow[d, phantom, "\overset{\kappa}{\Rightarrow}\,\cong"] \\
& \mathcal{D} \arrow[ul, phantom, shift right=3, "\cong\eta'"]
\end{tikzcd}$$

there exists a unique $\kappa : DF \to D'F$ such that $(\kappa \circ \Phi) \circ \eta = \eta'$.

However, these definitions are essentially equivalent.

(strict property (last time) $\Rightarrow$ 2-categorical property (today))

$\Leftarrow$ make Φ injective on objects and then take full subcategory on $\mathrm{Im}(\mathrm{Obj}\,\Phi)$

If we have a derived category $\Phi: \mathcal{C} \to D\mathcal{C}$ then it is unique up to $\cong$ (in the strict definition) resp. equivalence of categories (in the 2-categorical definition),

HW1: Choose one of these statements and prove it.

Existence: A universal algebraic approach runs into set-theoretical problems.

We are interested in the case $\mathcal{C} = Top$, $\mathcal{E}$ = weak equivalences.

Theorem (Whitehead): (1) If X is any space, there exists a weak equivalence $\gamma_X : X' \xrightarrow{\sim} X$ where X' is a CW-complex.

(2) If $e : X \xrightarrow{\sim} Y$ is a weak equivalence and Z is a CW-complex then

$$[Z, e] : [Z, X] \xrightarrow{\cong} [Z, Y] \qquad (*)$$

is a bijection where $[Z, X] = \mathrm{Mor}_{hTop}(Z, X)$.

spaces, homotopy classes of maps.

Note: a refinement of (2): if e is only assumed to be an n-equivalence then $(*)$ is injective assuming $\dim Z < n$ and onto assuming $\dim Z = n$.

The consequence of Whitehead's Theorem: Using the 2-categorical definition, we can take

$$\boxed{DTop = \underbrace{(\text{CW-complexes, homotopy classes of maps})}_{\text{The full subcategory of } hTop \text{ on CW-complexes}}}$$

This is the goal. To attain it, we need some general discussion of derived categories, and the proof of Whitehead's Theorem.

Quick example why this is useful: I get a generalised cohomology theory on CW-cxes just from a sequence $Z_n, \underbrace{n \in \mathbb{Z}}_{\text{equivalently } n \in \mathbb{N}_0}$ of based spaces with maps

weak equivalence

$$Z_n \xrightarrow{\sim} \Omega Z_{n+1}:$$

$$E^n(X) = [X, Z_n]$$

X CW-complex!

as before $= \mathrm{Mor}_{h\mathrm{Top}}(X, Z_n)$ homotopy classes of maps!

Theorem: A weak equivalence induces an isomorphism in singular homology and cohomology (any coefficients).

Proof strategy: Use Whitehead Thm., make a CW-complex out of a singular chain.

<u>Derived category discussion</u>: The content of the Whitehead Theorem is called <u>co-localisation</u>. ($\mathcal{C}$ = hTop)

how to get from Top to hTop

turns out, inverting equivalences

identifies homotopic maps

We'll discuss
co-localisation and
localisation next time
(and their applications
to constructing a derived category)
and the Whitehead Theorem.

$$\begin{tikzcd}
 & X \arrow[dl, bend right, "\mathrm{Id}"'] \arrow[d, "(x,0)"] \arrow[dr, bend left, "f"] & \\
X & X\times[0,1] \arrow[l, "\mathrm{proj}"'] \arrow[r] & Y \\
 & X \arrow[ul, bend left, "\mathrm{Id}"] \arrow[u, "(x,1)"'] \arrow[ur, "g"'] &
\end{tikzcd}$$

Equivalences (actually $\simeq$ equivalences)