MATH 695 10/12/2022

General statement of $\underbrace{\text{Whitehead's theorem}}_{\text{becomes an axiom}}$ in a category $\mathcal{C}$ with subcategory $\mathcal{E}$ of "equivalences" (contains $\cong$, satisfies 2/3): colocalization (dual statement is called localisation).

An object $Z \in \operatorname{Obj} \mathcal{C}$ is called co-local (with respect to $\mathcal{E}$) when for every equivalence $e : X \to Y \in \mathcal{E}$

$$\operatorname{Mor}_{\mathcal{C}}(Z, e) : \operatorname{Mor}_{\mathcal{E}}(Z, X) \xrightarrow{\cong} \operatorname{Mor}_{\mathcal{C}}(Z, Y)$$

is a bijection.

We say that $\mathcal{C}$ (with $\mathcal{E}$) has **co-localisation** with respect to a class of objects $\mathcal{B} \subseteq \mathrm{Obj}\,\mathcal{C}$ if every object $Z \in \mathcal{B}$ is co-local and for every object $X \in \mathrm{Obj}\,\mathcal{C}$ there exists an equivalence $\gamma_X : X' \xrightarrow{\sim} X$ where $X' \in \mathcal{B}$.

Example: $h\mathrm{Top}$ has co-localisation with respect to $\mathcal{B} = \{\text{CW-complexes}\}$ } Whitehead's Theorem

($\mathcal{E}$ = weak equivalences)

Dual notion of *localisation*: We say that $Z \in \mathrm{Obj}\,\mathcal{C}$ is *local* (w.r. to $\mathcal{E}$) if for every equivalence $e : X \xrightarrow{\sim} Y \in \mathcal{E}$

$$\mathrm{Mor}_{\mathcal{C}}(e, Z) : \mathrm{Mor}_{\mathcal{C}}(Y, Z) \xrightarrow{\cong} \mathrm{Mor}_{\mathcal{C}}(X, Z)$$

We say $\mathcal{C}$ has *localisation* with respect to a class of objects $\mathcal{B} \subseteq \mathrm{Obj}\,\mathcal{C}$ if for every object $X \in \mathrm{Obj}\,\mathcal{C}$ there exists $X' \in \mathcal{B}$ and an equivalence $\gamma_X : X \xrightarrow{\sim} X' \in \mathcal{E}$, and every object of $\mathcal{B}$ is local.

Comment: It is not known if hTop has localisation. But usually more "combinatorially" defined categories do: homotopy categories of sheaves, simplicial sets, etc.

Also, DTop actually has localisation with respect to

$$\mathcal{E} = \{ e : X \to Y \mid E_* e : E_* X \xrightarrow{\cong} E_* Y \}$$

where E is a generalised homology theory. (Bousfield localisation)

Theorem: If $\mathcal{C}$ has co-localisation (or localisation) on a class of objects $\mathcal{B}$ (with respect to a class of equivalences $\mathcal{E}$) then the derived category $D\mathcal{C}$ $(= D_{\mathcal{E}}\mathcal{C})$ exists and is equivalent to the $\underbrace{\text{full subcategory}}_{\substack{\text{has all the applicable morphisms} \\ \text{(all morphisms between objects in } \mathcal{B}\text{).}}}$ of $\mathcal{C}$ on $\mathcal{B}$.

Proof (the cases are symmetrical, let's discuss co-localisation - closer to the current context). $D = D_{\mathcal{E}}\mathcal{E}$

$$\mathrm{Obj}\, D := \mathrm{Obj}\, \mathcal{E}$$

$$\mathrm{Mor}_D(X,Y) := \mathrm{Mor}_{\mathcal{E}}(X',Y') \qquad (\gamma_X : X' \xrightarrow{\sim} X,\ X' \in \mathcal{B}).$$

Automatically a category. ? $\Phi : \mathcal{E} \to D$ identity on objects $f : X \to Y$

$$\begin{tikzcd}
X' \arrow[r, dashed, "\Phi f"] \arrow[d, "\sim\gamma_X"'] \arrow[dr] & Y' \arrow[d, "\gamma_Y\sim"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

$\exists!$

$\mathrm{Mor}_{\mathcal{E}}(X', \gamma_Y)$ is $\cong$ since X' is co-local.

Automatically a functor.

Showing that if $f \in \mathcal{E}$, Φf is an isomorphism:

$$\begin{tikzcd}
X' \arrow[r, "\Phi f"] \arrow[d, "\sim\gamma_X"'] & Y' \arrow[d, dashed, "\gamma_Y \sim"] \\
X \arrow[r, "f", "\sim"'] & Y
\end{tikzcd}$$

2/3 $\Rightarrow$ Φf is an equivalence.

(HW2) Prove that an equivalence between objects of $\mathcal{D}$ is an $\cong$.

Strict universality:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\Phi"] \arrow[dr, bend right, "F"'] & \mathcal{D} \arrow[d, dashed, "!"] \\
 & \mathcal{D}'
\end{tikzcd}$$

claimed $\mathcal{D}_{\mathcal{E}}\mathcal{C}$

$\mathcal{E} \longmapsto \cong$

apply F

$$\begin{tikzcd}
X' \arrow[r, "\Phi f"] \arrow[d, "\sim\gamma_X"'] & Y' \arrow[d, "\gamma_Y\sim"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
FX' \arrow[r, dashed, "F\Phi f"] \arrow[d, "\cong F\gamma_X"'] & FY' \arrow[d, "\cong F\gamma_Y"] \\
FX \arrow[r, "Ff"'] & FY
\end{tikzcd}$$

γ is the required equivalence of categories. $\square$

($WOLOG$, $\gamma : X' \xrightarrow[\mathrm{Id}]{} X$ if $X \in \mathcal{B}$)

By universality, the construction does not depend on the choice.

We prove that CW-complexes are co-local in hTop first.

This uses something called HELP (the homotopy extension and lifting property).

Reference (I.K., Kwz: Introduction to Algebraic Geometry)