MATH 695 10/14/2022

PROVING THE WHITEHEAD THEOREM

① Proving that CW-complexes are co-local in hTop:

If $e: X \xrightarrow{\sim} Y$ is a weak equivalence and Z is a CW-complex then

$$\mathrm{Mor}_{hTop}(Z,e) : \mathrm{Mor}_{hTop}(Z,X) \xrightarrow{\cong} \mathrm{Mor}_{hTop}(Z,Y).$$

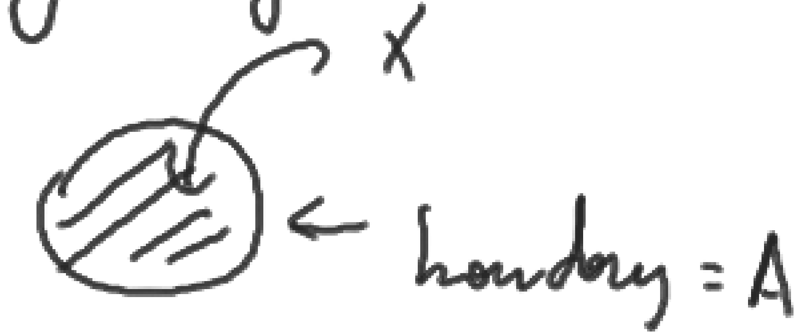
The key point in the proof is the Homotopy Extension and Lifting Property (HELP)

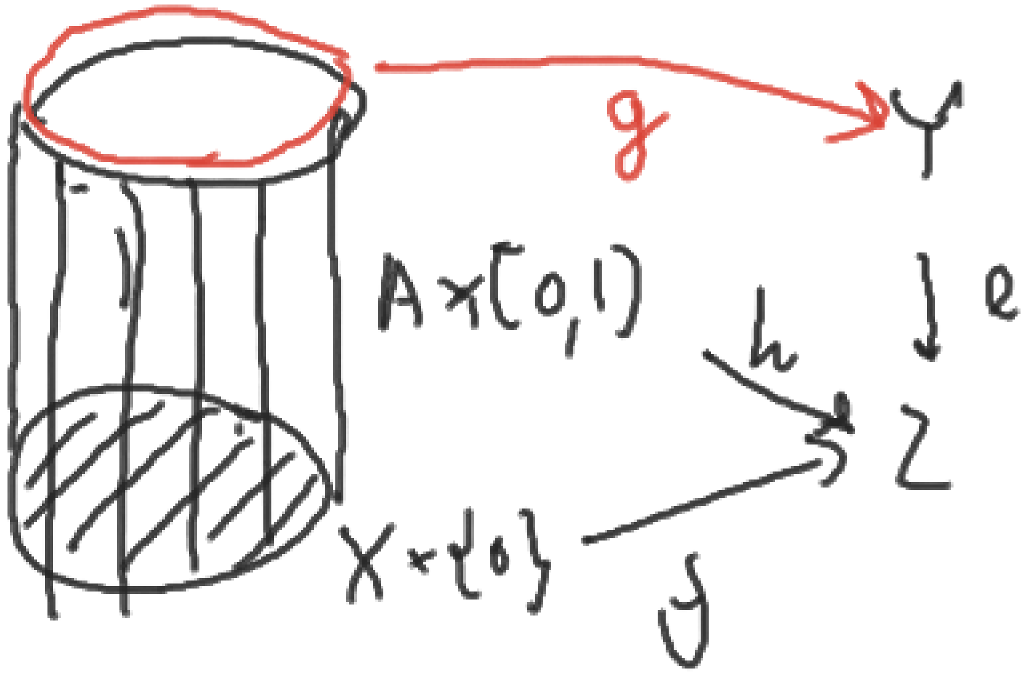
We say that a map $e : Y \longrightarrow Z$ satisfies HELP with respect to a pair (X, A) if the following diagram completes:

strictly, not just up to $\simeq$

$$
\begin{tikzcd}
A \arrow[rr, "{(a,0)}"] \arrow[dd, hook, "\subseteq"'] & & A \times [0,1] \arrow[dl, "h"'] \arrow[dd, hook, "\subseteq"] & & A \arrow[ll, "{(a,1)}"'] \arrow[dl, "g"] \arrow[dd, hook, "\subseteq"] \\
& Z & & Y \arrow[ll, "e"'] & \\
X \arrow[ur, "f"] \arrow[rr, "{(x,0)}"'] & & X \times [0,1] \arrow[ul, dashed, "H"'] & & X \arrow[ll, "{(x,1)}"] \arrow[ul, dashed, "G"']
\end{tikzcd}
$$

The "garbage can lemma"





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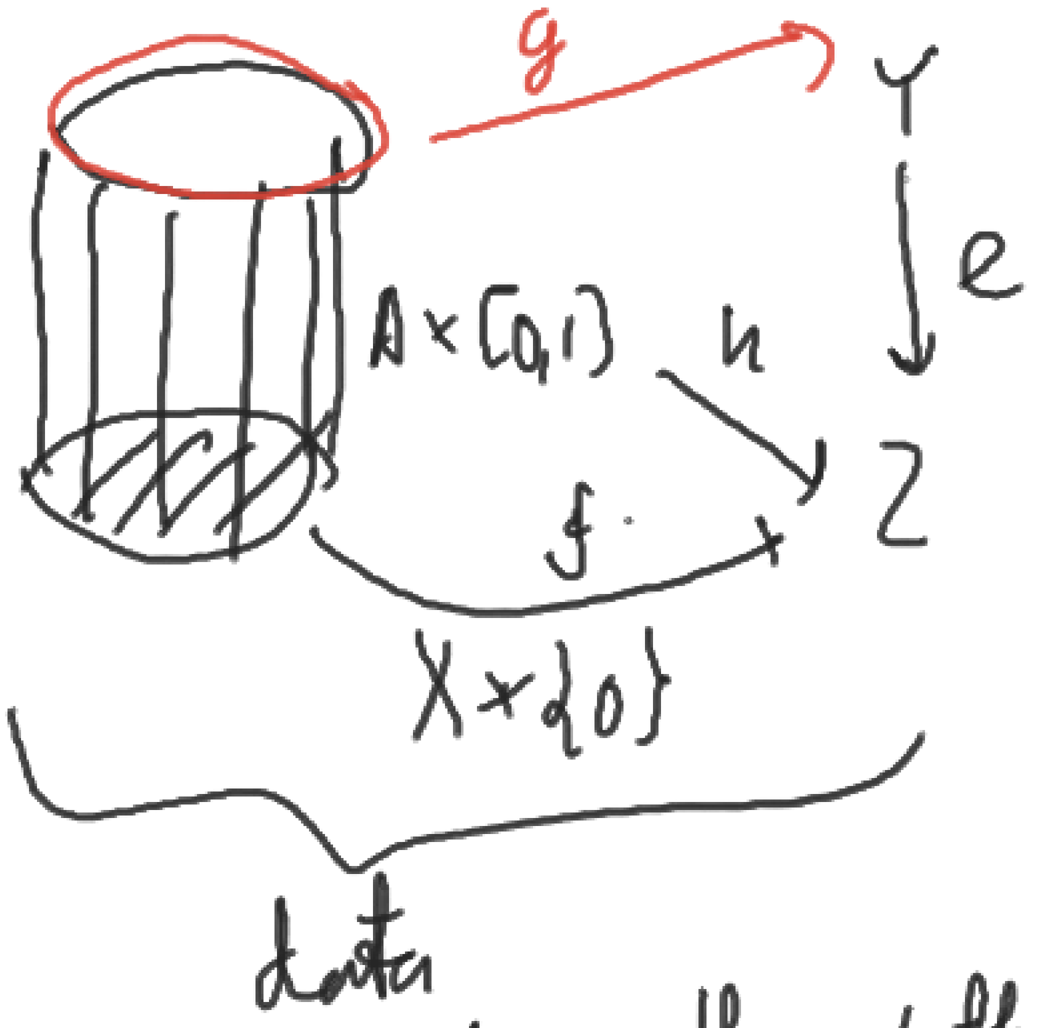
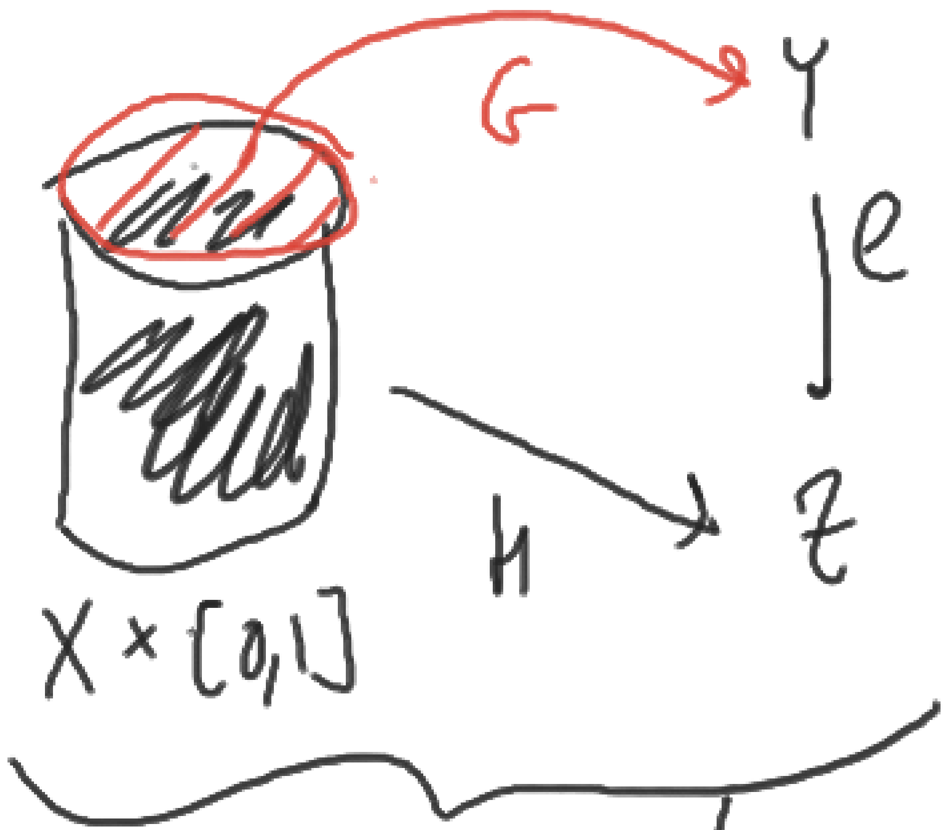
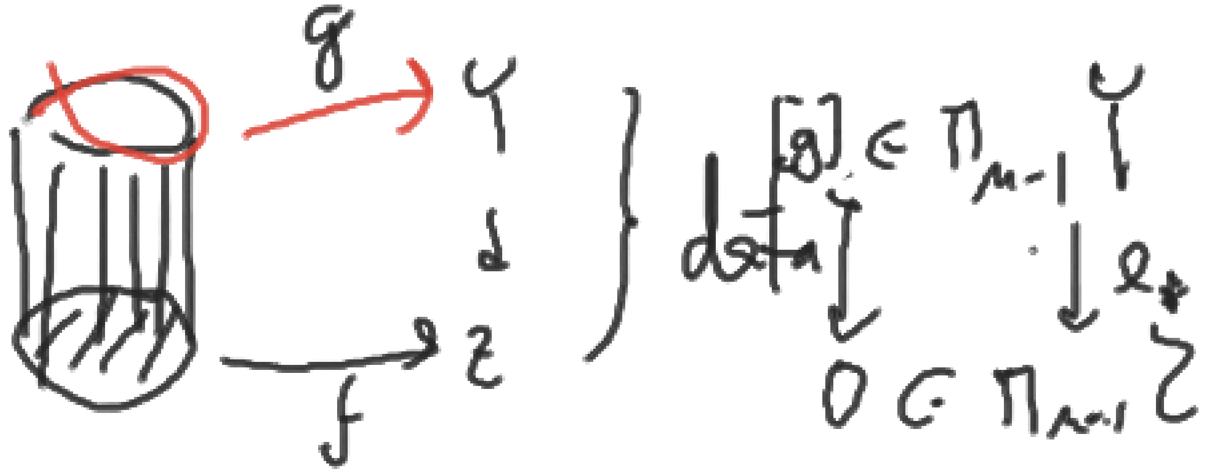


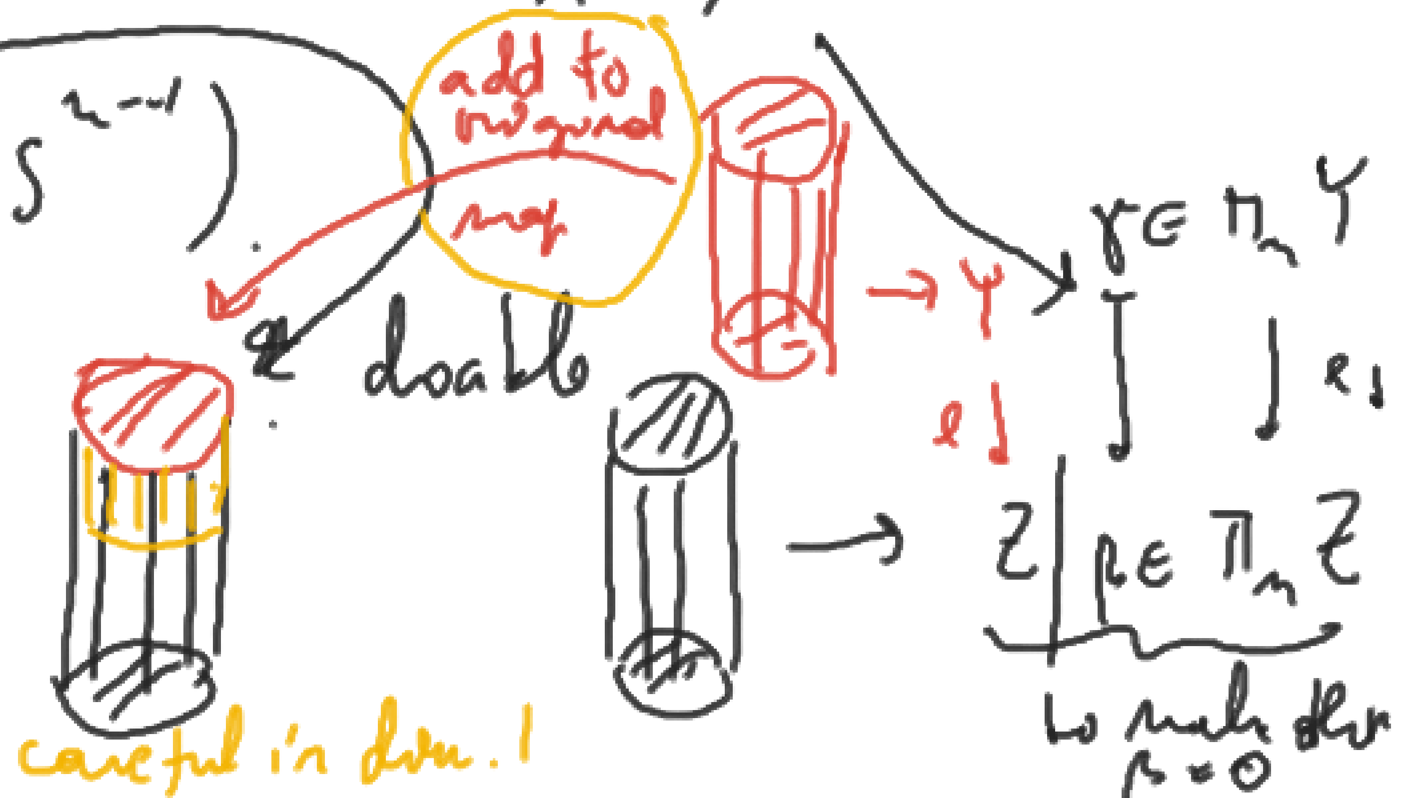


The statement of HELP

(everything path-connected)

<u>The HELP lemma</u>: If $\pi_{n-1}(e)$ is injective and $\pi_n(e)$ is onto then e satisfies HELP for $(X,A) = (D^n, S^{n-1})$





HELP lemma 2: If $e: Y \to Z$ is an n-equivalence (resp. a weak equivalence) then it satisfies HELP with respect to CW-pairs of $\underbrace{\text{dimension} \leq n}_{\text{cells in dim} \leq n}$ (resp. all CW-pairs).

Proof: Induction by cells. $\square$

Proving that CW-complexes are co-local in hTop

(refinement: If $e: X \to Y$ is an n-equivalence then

$$\mathrm{Mor}_{hTop}(Z,e): \mathrm{Mor}_{hTop}(Z,X) \to \mathrm{Mor}_{hTop}(Z,Y)$$

is a bijection when Z is a CW-complex of dimension $< n$ and onto when Z is a CW-complex of dim. n).

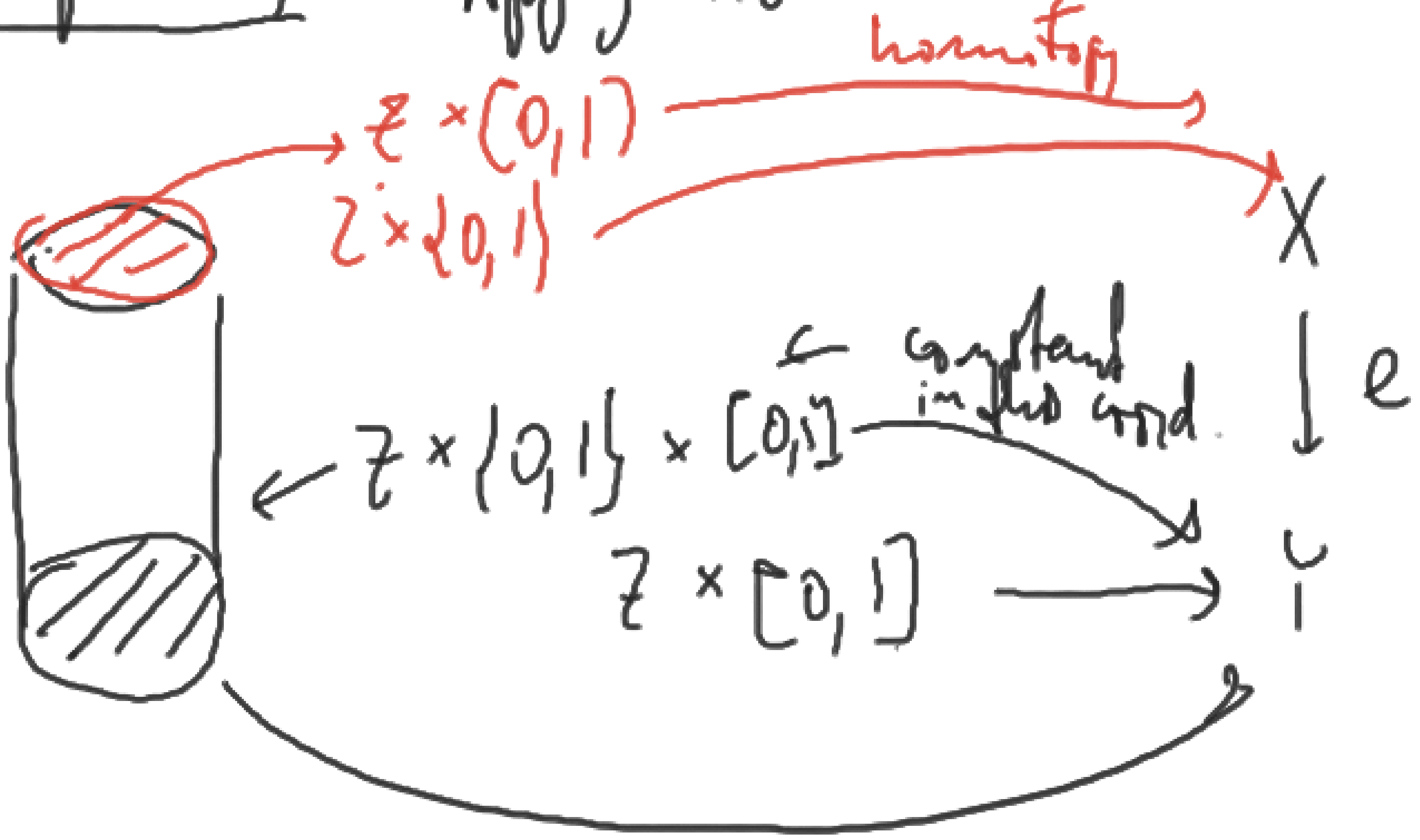
Given one $e : X \to Y$

$$\mathrm{Mor}_{hTop}(Z,e) : \mathrm{Mor}_{hTop}(Z,X) \longrightarrow \mathrm{Mor}_{hTop}(Z,Y)$$

<u>Surjectivity</u>: Apply HELP to the CW-pair $(Z, \emptyset)$

<u>Injectivity</u>: Apply HELP to the CW-pair $(Z \times [0,1], Z \times \{0,1\})$.

dimension goes up by 1



$\square$

<u>Cell approximation of maps</u>: A map $f: X \to Y$ where X, Y are CW-complexes is called <u>cell</u> if $f(X_n) \subseteq Y_n$.
(Note: cell maps obviously induce a homomorphism in cell homology) (so) ← contravariant

<u>Theorem</u>: If $f: X \to Y$ is a map of CW-complexes, then f is homotopic to a cell map.

Proof sketch: One cell at a time.. $\square$

HW ③ Prove that a <u>cell complex</u> is co-local in hTop. | all HW due Wed 10/19.

(Assuming colocalization by CW-complexes, this implies that every cell complex is homotopy equivalent to a CW-complex.)