MATH 695 10/19/2022

<u>Last time</u>: Let $f: X \to Y$ where X, Y are CW-complexes. Then there exists a cell map $g: X \to Y$ such that $f \simeq g$.

$\underbrace{\text{cell map } g: X \to Y}_{g(X_n) \subseteq Y_n}$

cell filtration

Also true for CW-pairs $f: (X, Z) \to (Y, T)$
Can assume $f|_Z = g|_Z$.

<u>Proposition</u>: Let X be a CW-complex. Then the inclusion $X_n \subseteq X$ is an n-equivalence.

<u>Proof</u>: WOLOG, connected.

$\pi_i X_n \to \pi_i X$. $\quad (S^i, *) \to (X, *)$ more precisely

By cell approximation, $i \leq n$, approximate $S^i \to X$ by a cell map.

Similarly, for injectivity:

$$(S^i \times [0,1],\ (S^i \times \{0,1\}) \cup (* \times [0,1])) \longrightarrow (X, X_n).$$

The restrictions to $S^i \times \{0\}$, $S^i \times \{1\}$ represent elements of $\pi_i X_n$ which become equal in $\pi_i X$.

Apply cellular approximation for pairs (need $i+1 \leq n$). $\square$

CW - approximation of spaces (a part of co-localisation):

Let X be a space. We construct, by induction, an n-equivalence

$$f_n : X'_n \to X$$

$n=0$: Put one point in each path-component. Suppose it is done for n.

(WOLOG $\underbrace{X \text{ path-connected}}_{\text{can assume a base point } * \in X_0}$ - otherwise one path-component at a time)

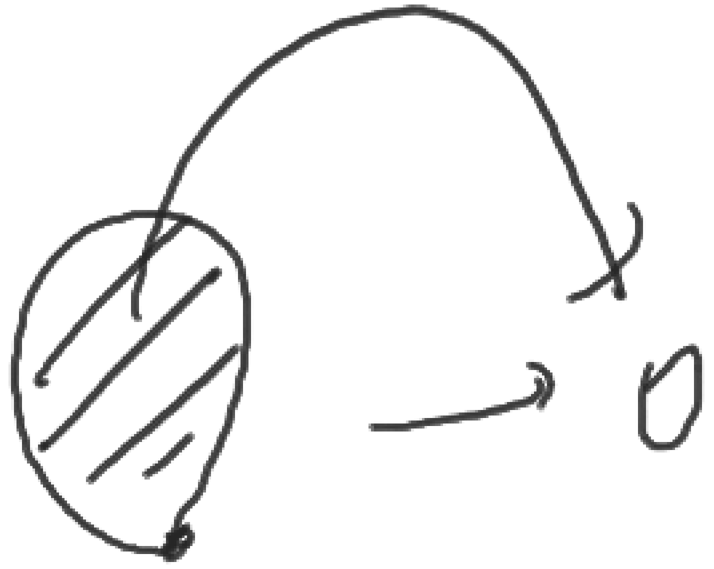
suffices to test π_i conditions at the base point.

$\exists\, I_{n+1}$ $\quad f'_n : X'_n \vee \bigvee_{I_{n+1}} S^{n+1} \longrightarrow X$ $\qquad X''_n := X'_n \vee \bigvee_{I_{n+1}} S^{n+1}$

where $f'_n | X'_n = f_n$ and f'_n is <u>onto</u> on π_{n+1}.

Construct X_{n+1} by attaching an $(n+1)$-cell to a representative of every element of

$$\operatorname{Ker}\left(\pi_n f_n' : \pi_n X_n'' \longrightarrow \pi_n X\right)$$



(note that $(X_n'')_n = X_n'$). $\square$

Less elaborate argument only giving $\underbrace{\text{cell approximation}}_{\text{by a cell complex.}}$:

Step 1: Attach cells to make all π_* onto.

Subsequent steps: Always attach cells to kill the kernel.

$\bigcup_n$ step n OK.

Hurewicz map: Let X be path-connected based.

Then we have a map

$$h : \underbrace{\pi_n X}_{\alpha : S^n \to X} \longrightarrow H_n(X; \mathbb{Z}) \qquad \text{Hurewicz map}$$

$n > 0$

Choose, once and for all, a generator $u \in H_n(S^n; \mathbb{Z}) = \mathbb{Z}$. Set

$$h[\alpha] := (H_n\alpha)(u)$$

$$\begin{array}{rcl} H_n\alpha : H_n(S^n; \mathbb{Z}) & \longrightarrow & H_n(X; \mathbb{Z}) \\ \| & & \\ \mathbb{Z} & & \\ \Psi & & \\ u & \longmapsto & h(\alpha). \end{array}$$

Homomorphism.

__Hurewicz Theorem__: If $\pi_i(X) = 0$ for $i < n$ (we say X is $(n-1)$-connected) then

$$h : \pi_n(X) \to H_n(X; \mathbb{Z})$$

is an isomorphism if $n > 1$ and abelianization if $n = 1$.

$G/[G,G]$

$(n+1)$-equivalence

Proof: We construct an $(n+1)$-approximation $X'_{n+1} \longrightarrow X$ (This preserves π_n, H_n.) Let

$$X'_n = \bigvee_{I_n} S^n$$

I_n is a set of generators of $\pi_n X$.

$$X''_n = \bigvee_{I_n} S^n \vee \bigcup_{I_{n+1}} S^{n+1}$$

generators of π_{n+1}, along for the ride.

Now $X'_{n+1} = X''_n \cup (n+1)$-cells along the relations in $\pi_n(X)$.

Why does X'_{n+1} have the correct H_n $\left(= \underset{n>1}{\pi_n} \text{ resp. } \pi_1/[\pi_1,\pi_1]\right)$?

It has the same generators and relations! $\square$

$$S^n \to S^n \vee S^n$$
$$u \mapsto (u,u)$$

(Note: This uses the fact that weak equivalence preserves singular homology.)

Construction of Eilenberg-MacLane spaces:

I construct a CW-complex X such that $\pi_n(X) = G$ — discrete group, abelian if $n>1$

$\underbrace{K(G,n)}$. $\quad \pi_i(X) = 0 \quad i \neq n$.

$n > 0$.

Construct X_{n+1} connected n-dim. CW-cx with one 0-cell, no cells in dimensions $1 \leq i < n$, n-cells: generators of G and $(n+1)$-cells are the relations of G. In each step $k \geq n+1$, attach $(k+1)$-cells to X_k to kill π_k.

HW1: Prove that if X, Y are CW-complexes of type $K(G,n)$ then $X \simeq Y$.

Note: In particular,

weak equivalence

$$K(G,n) \xrightarrow{\sim} \underbrace{\Omega K(G,n+1)}_{\substack{\text{has the homotopy groups of } K(G,n), \\ \text{but isn't necessarily a CW complex.}}}$$

Construct singular cohomology of CW-cxes.