MATH 695 10/21/2022

Ordinary homology and cohomology have a strong relationship between them.

$$H^n(X;G) \neq \mathrm{Hom}(H_n(X;\mathbb{Z}),G)$$

$$H_n(X;G) \neq H_n(X;\mathbb{Z})\otimes G$$

in general (although sometimes yes).

<u>Example</u>: $H_n(\mathbb{R}P^k;\mathbb{Z})$ using cell homology:

$$H_n\Big(\mathbb{Z} \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}\Big)$$

dim: k $\qquad\qquad\qquad 0$

$$\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \; \cdots \; \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

$\mathbb{Z} \xrightarrow{2} \mathbb{Z}$

Homology: k even: $0 \quad \mathbb{Z}/2 \; \cdots \; 0 \quad \mathbb{Z}/2 \quad 0 \quad \mathbb{Z}/2 \quad \mathbb{Z}$

$\mathbb{Z} \xrightarrow{0} \mathbb{Z}$

k odd: $\mathbb{Z} \quad 0$

$H_n(\mathbb{RP}^k; \mathbb{Z})$
$= \mathbb{Z}$ if $n=0$ or $n=k$ odd
$\mathbb{Z}/2$ $\quad 0<n<k$ odd
0 else

$\otimes \mathbb{Z}/2$: $\quad \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2$

$H^n(\mathbb{RP}^k; \mathbb{Z}/2) \cong H_n(\mathbb{RP}^k; \mathbb{Z}/2) =$
$= \mathbb{Z}/2 \quad 0 \le n \le k$
0 else

0 k odd

2 k even

$$\mathbb{Z} \leftarrow \mathbb{Z} \leftarrow \mathbb{Z} \qquad \leftarrow \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z}$$

$H^n(\mathbb{RP}^k; \mathbb{Z}) = \mathbb{Z}$ if $n=0$ or $n=k$ odd
$\mathbb{Z}/2$ if $0<n\le k$ even
0 else.

All the information is in the chain complex (with coeffs. in $\mathbb{Z}$).

$C(X) \leftarrow$ chain cx. of free abelian groups.

Given a chain cx. of free abelian groups:

$$
\begin{tikzcd}
\arrow[r] & C_n \arrow[r, "d"] & C_{n-1} \arrow[r, "d"] & C_{n-2} \arrow[r] & \cdots \\
\cdots & Z_n \oplus B_{n-1} \arrow[u, "\cong" ', "\oplus s"] & Z_{n-1} \oplus B_{n-2} \arrow[u, "\cong"'] & Z_{n-2} \oplus B_{n-3} \arrow[u, dash, "\cong"'] & \cdots
\end{tikzcd}
$$

$$B_{n-1} \overset{\subseteq}{\nearrow} Z_{n-1}, \qquad B_{n-2} \overset{\subseteq}{\nearrow} Z_{n-2}$$

$Z_n = \mathrm{Ker}(d : C_n \to C_{n-1})$

$B_n = \mathrm{Im}(d : C_{n+1} \to C_n)$

$$
\begin{tikzcd}
0 \arrow[r] & Z_n \arrow[r] & C_n \arrow[r] & B_{n-1} \arrow[r] \arrow[l, dashed, bend left, "s"] & 0
\end{tikzcd}
$$

a subgroup of a free abelian group is free abelian (works for R-modules $R = PID$)

Let H be any abelian group. Then we have a short exact sequence:

$$0 \to B \xrightarrow{i} Z \to H \to 0$$

B: $\therefore$ free abelian; Z: free abelian

Consider the 2-stage chain cx:

$$\mathcal{H}: \cdots 0 \quad B \xrightarrow{i} Z \quad 0 \cdots$$

dim degree: 1   0

<u>Proposition</u>: Suppose $C = (\cdots \to C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} \cdots)$ is a chain complex of free abelian groups, $H_n = H_n C$. Then

$$C \cong \bigoplus_{n \in \mathbb{Z}} \mathcal{H}_n [n] \qquad \square$$

(Funny point about the notation: $\mathcal{H}$ is not determined up to $\cong$ by H.)

should we be doing homotopy theory of chain complexes?

For the chain complex $\mathcal{H}$, and an abelian group G, I can consider the chain complex $\mathcal{H} \otimes G$ and its homology:

$$\cdots \quad 0 \qquad B\otimes G \xrightarrow{i\otimes G} Z\otimes G \qquad 0 \quad \cdots$$

$$\operatorname{Ker}(i\otimes G) =: \operatorname{Tor}_1^{\mathbb{Z}}(H,G), \qquad Z\otimes G/\operatorname{Im} i\otimes G \cong Z/B \otimes G \cong H\otimes G = \operatorname{Tor}_0^{\mathbb{Z}}(H,G)$$

"we don't know the answer so let's give it a name!"

Cohomology of $\operatorname{Hom}(\mathcal{H},G)$:

$$0 \qquad \operatorname{Hom}(B,G) \xleftarrow{\operatorname{Hom}(i,G)} \operatorname{Hom}(Z,G) \qquad 0$$

$$\operatorname{Hom}(B,G)/\operatorname{Im}(\operatorname{Hom}(i,G)) =: \operatorname{Ext}^1_{\mathbb{Z}}(H,G) \qquad \operatorname{Hom}(H,G)$$

$\otimes$, $\mathrm{Tor}_1^{\mathbb{Z}}$ commute with $\oplus$

Hom, $\mathrm{Ext}^1_{\mathbb{Z}}$ take $\oplus$ in first variable to $\prod$.

(direct sum of 2 still preserved).

HW 2 Compute $\mathrm{Tor}_1^{\mathbb{Z}}(H, G)$, $\mathrm{Ext}^1_{\mathbb{Z}}(H, G)$ where H, G are cyclic ($\mathbb{Z}$ or $\mathbb{Z}/n$).

Independence of the choice of $\mathcal{H}$?

$\Downarrow$

homotopy theory of chain complexes.

Theorem: Let X be a space. Universal coefficient theorem

$$H_n(X;G) \cong H_n(X;\mathbb{Z}) \otimes G \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(X;\mathbb{Z}), G)$$

$$H^n(X;G) \cong \mathrm{Hom}(H_n(X;\mathbb{Z}), G) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(X;\mathbb{Z}), G)$$

Proof: First summand: $H_0(\underset{\overset{\shortparallel}{H_n \otimes G}}{\mathcal{H}_n \otimes G}) = H_n(\mathcal{H}_n[n] \otimes G)$

Second summand: $H_1(\mathcal{H}_{n-1} \otimes G) = H_n(\mathcal{H}_{n-1}[n] \otimes G)$

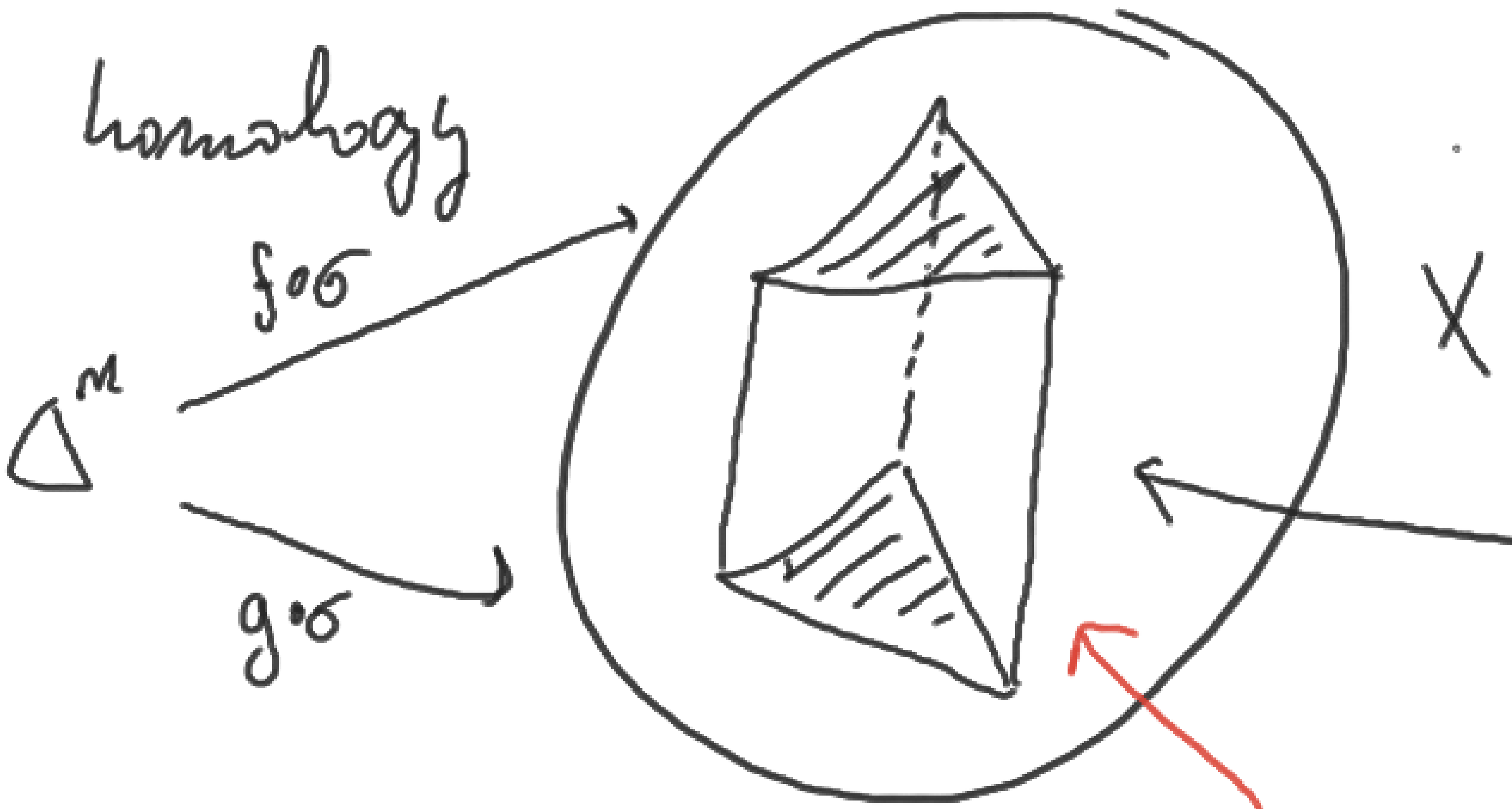
$$\shortparallel$$
$$\mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}, G).$$

Similarly for cohomology. $\square$

Note: $\oplus$ depends on choice of splitting $\Rightarrow$ not natural.

Proving the homotopy axiom for singular homology involves introducing chain homotopy and proving that chain-homotopic chain maps induce the same homomorphism in homology.

singular homology



Suppose $f \simeq g$

$h\circ(\sigma\times[0,1]) : \underbrace{\Delta^n \times [0,1]}_{\text{prism}} \to X$

you need to triangulate the prism

Assume this: $f_* - g_* = dh_* + h_* d.$

maps induced on chains

Definition: If $f, g: C \to D$ are chain maps then a chain homotopy between them is a sequence of homomorphisms

$h = (h_n : C_n \to D_{n+1})$

so that

$$f - g = dh + hd$$

$$f_n - g_n = d h_n + h_{n-1} d.$$