MATH 695 10/24/2022

Homotopy theory of chain complexes

d is a map of R-modules

of R-modules, R commutative ring

M, N R-modules $\Rightarrow$ $M \otimes_R N$ is an R-module

commutative, associative, unital ACU

up to natural $\cong$, coherence diagrams

Chain complexes of R-modules also have a ACU $\otimes$ product

$$C = \left(\to C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} C_{n-2} \to \cdots \right)$$

$$\square$$

$$D = \left(\to D_n \to D_{n-1} \xrightarrow{d} D_{n-2} \to \cdots \right)$$

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
\cdots \arrow[r] & C_k \otimes_R D_\ell \arrow[r, "d\otimes 1"] \arrow[d, "1\otimes d"] & C_{k-1} \otimes_R D_\ell \arrow[r, "d\otimes 1"] \arrow[d, "1\otimes d"] & C_{k-2} \otimes_R D_\ell \arrow[r] \arrow[d, "1\otimes d"] & \cdots \\
\cdots \arrow[r] & C_k \otimes_R D_{\ell-1} \arrow[r, "d\otimes 1"] \arrow[d] & C_{k-1} \otimes_R D_{\ell-1} \arrow[r] \arrow[d] & C_{k-1} \otimes_R D_{\ell-1} \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

diagrams commute

horizontally, vertically: chain complexes

This is called a <u>double chain complex</u>: $C = (C_{mn}, \partial, \delta)$

$$
\begin{tikzcd}
 & \arrow[d] & \arrow[d] & \\
\cdots \arrow[r] & C_{mn} \arrow[r, "\partial"] \arrow[d, "\delta"] & C_{m-1,n} \arrow[r] \arrow[d, "\delta"] & \cdots \\
\cdots \arrow[r] & C_{m,n-1} \arrow[r, "\partial"] \arrow[d] & C_{m-1,n-1} \arrow[r] \arrow[d] & {} \\
 & {} & {} &
\end{tikzcd}
$$

$$\partial\partial = 0$$

$$\delta\delta = 0$$

$$\delta\partial = \partial\delta$$

$|\;|$ usually omitted

Totalisation of a double chain complex:
(a single chain complex)

$$|C|_n = \left(\to \bigoplus_{k+l=n} C_{kl} \to \bigoplus_{k'+l'=n-1} C_{k'l'} \to \right)$$

$$d = \partial + (-1)^k \delta$$

We denote for chain complex of R-modules C, D

$$C \otimes_R D := |C \otimes_R D|$$

double chain complex.

This is an ACU operation on the category of chain cxes of R-modules.

Commutativity:

$$C \otimes_R D \xrightarrow[\tau]{\cong} D \otimes_R C$$

$x \otimes y \in C_k \otimes_R D_l$

$\tau(x \otimes y) := (-1)^{kl} y \otimes x$

<u>Note</u>: If X, Y are <u>CW-cxes</u> (in the compactly generated setting - May: A concise course -r)

$X \times Y$ is a CW-complex.

$$C^{cell}(X \times Y) \cong C^{cell}(X) \otimes_{\mathbb{Z}} C^{cell}(Y).$$

For singular homology, $C(X\times Y) \not\simeq C(X)\otimes_{\mathbb{Z}} C(Y)$
(except maybe if X or Y is a point)

Eilenberg-Zilber Theorem (Munkres)

$$C(X\times Y) \underset{\varphi}{\overset{\psi}{\rightleftarrows}} C(X)\otimes_{\mathbb{Z}} C(Y)$$

chain homotopy

$$\underbrace{\varphi,\psi \text{ natural},\ \psi\varphi \simeq \mathrm{Id},\ \varphi\psi \simeq \mathrm{Id}}_{\text{chain homotopy equivalence}}$$

If C, D are chain complexes of free abelian groups, we can ask what is $H_*(C\otimes_{\mathbb{Z}} D)$ in terms of $H_*(C)$, $H_*(D)$?

We can answer this question if we can answer it for $\tilde{\mathcal{X}}$

2-stage complexes:

dim. degree 0

$$\mathcal{X}: \quad B \xrightarrow{\subseteq} Z \qquad\qquad Z/B = H$$

$$\mathcal{X}': \quad B' \xrightarrow{\subseteq} Z' \qquad\qquad Z'/B' = H'$$

$$\boxed{0 \to B \to Z \to H \to 0}$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ 1 & 0 & -1 \end{array} \leftarrow \text{degree}$$

Homology of $\mathcal{X} \otimes \mathcal{X}'$?

$$0 \to H[-1] \to \tilde{\mathcal{X}} \to \mathcal{X} \to 0$$

$$0 \to H[-1] \otimes_{\mathbb{Z}} \mathcal{X}' \to \tilde{\mathcal{X}} \otimes_{\mathbb{Z}} \mathcal{X}' \to \mathcal{X} \otimes \mathcal{X}' \to 0$$

Homology 0

$$\begin{tikzcd} \mathcal{C} & 0 \arrow[r] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & 0 \\ \mathcal{B} & 0 \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & 0 \end{tikzcd}$$

rows exact

$$0 \to \mathcal{B} \to \tilde{\mathcal{X}} \otimes \mathcal{X}' \to \mathcal{C} \to 0$$

0 homology.

$$\therefore H_n(\mathcal{X} \otimes_{\mathbb{Z}} \mathcal{X}') = \mathrm{Tor}_n^{\mathbb{Z}}(H, H').$$

∴ **Künneth Theorem**: C, D chain cxes of abelian groups: (X, Y spaces)

$$H_n(C \otimes_{\mathbb{Z}} D) = \bigoplus_{k+l=n} H_k(C) \otimes_{\mathbb{Z}} H_l(D) \oplus \bigoplus_{k+l=n-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_k C, H_l D)$$

$$H_n(X \times Y; \mathbb{Z}) = \bigoplus_{k+l=n} H_k(X;\mathbb{Z}) \otimes H_l(Y;\mathbb{Z}) \oplus \bigoplus_{k+l=n-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_k(X,\mathbb{Z}), H_l(Y,\mathbb{Z}))$$

$\square$

HW ① Calculate $H_i(\mathbb{R}P^n \times \mathbb{R}P^m; \mathbb{Z})$.