MATH 695 10/26/2022

Side note: An abelian category = a category with finite limits and colimits with 0 (initial obj $\cong$ terminal object) every epimorphism is a cokernel and any monomorphism is a kernel.

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epimorphism = categorical model of surjection: $\forall g, h : Y \to Z$:

$$X \xrightarrow{f} Y \underset{h}{\overset{g}{\rightrightarrows}} Z \qquad g \circ f = h \circ f \Rightarrow g = h.$$

(equivalently: $\mathrm{Coeq}(g \circ f, h \circ f) = \mathrm{Coeq}(g, h)$.)

If a functor preserves (finite) colimits (such as a left adjoint), it also preserves epimorphisms).

A **cokernel** is a coequaliser with 0: $f: X \to Y$

$$\mathrm{Coker}\, f = \mathrm{Coeq}\left(X \overset{f}{\underset{0}{\rightrightarrows}} Y \right)$$

$$0 : X \to Y \;=\; X \to 0 \to Y$$

initial = terminal

One can prove that an abelian category is **additive**.

Pre-additive category $\mathcal{C}$: $\mathrm{Mor}_{\mathcal{C}}(X, Y)$ is given the structure of an abelian group, composition is bilinear.

Bi-product $X \oplus Y$: characterised by equations

$$X \rightleftarrows X \oplus Y \leftrightarrows Y$$

product = coproduct

.... SES + splittings

If always $\exists$, $\mathcal{C}$ is called **additive**

We have a complete $\underbrace{\text{theory of exact sequences, homology}}_{\text{additive homological algebra}}$

in an abelian category. Additive categories are a lot more general.

(K., Kir: Intro to Alg. Geom.)

left (or right)
R-modules (R ring) form an abelian category.

"internal
Tensor products" (associative, comm., unital), bilinear

on an abelian category: Tensor category (Terminology varies)

R commutative ring $\Rightarrow$ R-modules form a tensor category ($\otimes_R$).

Back to homotopy thy of chain complexes:
(can be set up in an abelian category $\mathcal{C}$).

The "unit interval":

cell chain complex of $[0,1]$

$$I : \left(\quad 0 \qquad \underset{\text{degree } 1}{\mathbb{Z}} \xrightarrow{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} \underset{\text{degree } 0}{\mathbb{Z} \oplus \mathbb{Z}} \qquad 0 \quad \cdots \right)$$

$$* = \quad 0 \quad \underset{\text{degree } 0}{\mathbb{Z}} \quad 0$$

$1 \cdot \cong \underline{0}$, $\quad 1 \cdot \cong \underline{1}$

$$* \underset{\underline{1}}{\overset{\underline{0}}{\rightrightarrows}} I$$

Ab^{fg} = full subcategory of Ab on f.g. abelian groups

Ab = (Abelian groups, homomorphisms). If $\mathcal{C}$ is an abelian category, there is always a canonical

$$\otimes : Ab \times \mathcal{C} \to \mathcal{C}$$

$\mathbb{Z}^n \otimes X = \bigoplus_n X$

pass to quotient

Chain complex analog of $X \times [0,1]$ for a space X.

C = chain complex in an abelian category $\mathcal{C}$:

$$I \otimes C$$

tensor product of chain cxes:

C_{n-1}
$\oplus$
C_n
$\oplus$
C_n

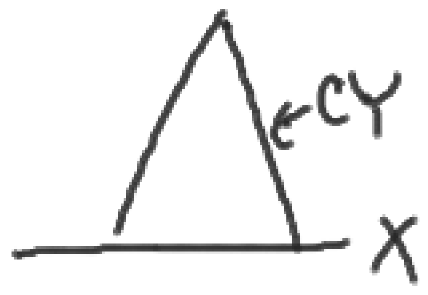
deg n

(Based) mapping cone: $f: C \to D$ a chain map

$$\operatorname{colim}\left(\begin{array}{ccc} *\otimes C & \longrightarrow & 0 \\ {\scriptstyle 1\otimes \mathrm{Id}}\big\downarrow & & \\ I\otimes C & & \\ {\scriptstyle 0\otimes\mathrm{Id}}\big\uparrow & & \\ \underbrace{*\otimes C}_{=C} & \xrightarrow{f} & D \end{array}\right) =: Cf$$

deg n: $D_n \oplus C_{n-1}$

$f: Y \to X$



HW2: ⓐ Give an explicit description of Cf for a chain map $f : C \to D$.

ⓑ Exhibit a short exact sequence of chain complexes

$$0 \to D \to Cf \to C[1] \to 0$$

↑ shift up by 1 in degree

(Cartan-Eilenberg: Homological Algebra)

∴ LES in homology: $f : C \to D$ chain map

$$\to H_nC \xrightarrow{H_nf} H_nD \to H_nCf \to H_{n-1}C \to H_{n-1}D \to$$

(note the analogy with π_n in spaces. One can also set up homology as "homotopy groups"

(*) $H_n C = \{ \mathcal{C}\text{-Chain}(\mathbb{Z}[n], C) / \text{chain} \simeq \}$

(just think of $\mathcal{C} = Ab$),

$\mathcal{C}\text{-Chain} : Ab\text{-Chain} \times \mathcal{C}\text{-Chain} \to \mathcal{C}$

$\mathrm{Hom} : Ab^{fin} \times \mathcal{C} \to \mathcal{C}$

is also defined when $\mathcal{C}$ is an abelian category

$\mathrm{Hom}(\mathbb{Z}^n, X) = \bigoplus_n X$

$\mathbb{Z}^n \xrightarrow{g} \mathbb{Z}^m \to G \to 0$

$\mathrm{Hom}(G, X) = \mathrm{Ker}(\mathrm{Hom}(g, X))$

Also, <u>chain homotopy</u> between chain maps $f, g : C \to D$ is equivalent to

$$h : I \otimes C \to D \qquad \begin{array}{l} f = h(\underline{0} \otimes Id_C) : C \to D \\ g = h(\underline{1} \otimes Id_C) : C \to D \end{array}$$

HW3 : Verify the alternative definition of chain homotopy for chain cxes of abelian groups.