MATH 695 10/28/2022

If $f: C \to D$ is a morphism in $\mathcal{A}$-Chain where $\mathcal{A}$ is an abelian category, then we have a long exact sequence

$$\cdots \to H_n C \xrightarrow{H_n f} H_n D \longrightarrow H_n Cf \longrightarrow H_{n-1} C \xrightarrow{H_{n-1} f} H_{n-1} D \to \cdots ,$$

We were able to get a close analogy with the long exact sequence of homotopy groups ($S^n \rightsquigarrow \mathbb{Z}[n] \in \text{Ab-Chain}$).

Note: in chain complexes, we are already in the $\underbrace{\text{based context}}_{\text{category with } 0}$

Differences:

- In chain complexes, the long exact sequence never ends (extends without end on both sides)
- For spaces, there was one LES for a cofibration and one for a fibration. So we could investigate this and define homotopy fibre in chain cxes.

$$Ff \longrightarrow C \xrightarrow{f} D$$

The concept of an abelian category is self-dual. If $\mathcal{A}$ is an abelian category, $\mathcal{A}^{op}$ is also an abelian category. So Ff is just Cf in $\mathcal{A}^{op}$-Chain.

Moreover,

HW4: In $\mathcal{A}$-Chain for an abelian category $\mathcal{A}$, $f: C \to D$,

$$Ff \cong Cf[-1].$$

Philosophically, this is <u>expected</u> since there is only one LES in homology (and not another dual one).

<u>Homotopical stability</u>

- $?[1]$ (shift) is an equivalence of categories
- Therefore, chain complexes are a priori not bounded below.

Note: chain complexes lose information. Homotopically stable information is measured by *generalized* (co)homology theory. Chain complexes only capture ordinary (co)homology (collapse of AHSS).

Homotopy theory in $\mathcal{A}$-Chain ($\mathcal{A}$ abelian category).

This means identifying a derived category. So we need a notion of an *equivalence*.

An *equivalence* of chain complexes is a chain map $f : C \to D$ which induces an isomorphism in homology. This is also called a *quasi-isomorphism*.

The derived category $D\mathcal{A}$ of an abelian category $\mathcal{A}$ (if one exists)
is the derived category with respect to quasi-isomorphisms.

Can we construct $D\mathcal{A} = D\mathcal{A}\text{-Chain}$ using (co)localisation?
Yes, under some assumptions.

Clue: Free resolutions: R-module M, R-chain complex

$C = (\to C_n \to C_{n-1} \to \dots \to C_0)$

$H_0 C \overset{\cong}{\leftarrow} M$, $H_i C = 0$ for $i > 0$, C_i are free R-modules. (Who the $\mathcal{H}$ for an abel category H)

not a notion in an abelian category.

A property of free R-modules P: Any short exact sequence

$$0 \to M \to N \xrightarrow{p} P \to 0$$

splits: $\exists s \quad ps = Id_P$.

This property is called projectivity.

P is projective

The dual notion, turning around arrows, is called injective.

An abelian category $\mathcal{A}$ has enough projectives if for every $M \in$ Object there exists an epimorphism $P \to M \to 0$ where P is projective.

(Dual notion: $\forall M \; \exists Q \quad 0 \to M \to Q$, Q injective is called enough injectives)

If we have enough projectives, we have projective resolutions:

$$\left.\begin{array}{l} 0 \to M_1 \to P_0 \to M \to 0 \\ 0 \to M_2 \to P_1 \to M_1 \to 0 \\ \quad\quad \vdots \end{array}\right\}$$

$$\boxed{\begin{array}{l} P : (\cdots \to P_2 \to P_1 \to P_0) \\ H_0 P \xrightarrow{\cong} M \\ H_i P = 0 \quad i > 0 \\ P \text{ projective} \end{array}}$$

a projective resolution of M

($\exists$ if enough projectives)

Dual notion: An injective resolution $(Q^0 \to Q^1 \to Q^2 \to \cdots\cdots)$ Q

$H^0 Q \xrightarrow{\cong} M$, $H^i Q = 0$ for $i > 0$ where Q are injective exist if enough injectives

Proposition: In $h\mathcal{A}$-Chain, projective resolutions are co-local and injective resolutions are local with respect to quasi-isomorphisms.

Not enough to claim that $h\mathcal{A}$-Chain has localisation or co-localisation: Resolutions only apply to $M[n]$ for $M \in \mathcal{A}$. We need "resolutions" of every $\mathcal{A}$-chain complex. We need an analog of cell objects (or co-cell objects) and the Whitehead Thm. (e.g. co-Whitehead theorem).

Not a priori bounded below (or above) so CW would need further discussion. ← possible, but we won't do it.

Defining <u>cell objects</u> is easy:

$$C = \operatorname{colim}\left(C_{(-1)} \to C_{(0)} \to C_{(1)} \to \cdots \right)$$

when $C_{(-1)} = 0$,

mapping cone

$$C_{(n+1)} = C\left(\varphi_n : P_{(n)} \to C_{(n)} \right)$$

where $P_{(n)}$ is a chain complex consisting of projective objects <u>with 0 differential</u>.

<u>Theorem</u> (algebraic Whitehead): If $\mathcal{A}$ has enough projectives and coproducts then $\mathrm{h}\mathcal{A}$-Chain has co-localisation by cell objects.