MATH 695 10/31/2022

Cell chain cx. in an abelian category: $C = \operatorname{colim} C_{(n)}$

$$C_{(-1)} \to C_{(0)} \to C_{(1)} \to \cdots$$

mapping cone

$C_{(-1)} = 0$ (cell pair: $C_{(-1)}$ is anything), $C_{(n+1)} = C\varphi_{(n)}$

$$\varphi_{(n)} : P_{(n)} \to C_{(n)}$$

where $P_{(n)}$ is a chain complex of projective objects with 0 differential.

(notion of a co-cell chain cx. vs dual)

<u>Theorem</u>: Cell chain complexes in an abelian category $\mathcal{A}$ are co-local in hot-Chain. Co-cell chain complexes are local in hot-Chain.

<u>Proof</u>: One approach is <u>exactly analogous</u> to the case of spaces: HGLP. $\square$

Note: it works more generally. For example, a Differential Graded Algebra is a chain cx A of R-modules (R comm. ring) with a multiplication $A \otimes_R A \to A$ which is associative and unital. The co-local part of the Theorem is true for the category of $^{(left)}$DG-modules over A. Cell objects: $\varphi_{(n)}: P_{(n)} \to C_{(n)}$ (K., May ...)

$C_{(n+1)} = C\varphi_{(n)}$

$\bigoplus$ free DG-A-modules

$A \otimes_R M \to M$

associative

unital

Existence of (co)-localisation: If $\mathcal{A}$ is an abelian category with enough projectives and coproducts then for every chain complex C in $\mathcal{A}$ $\forall M$ object $\exists P \twoheadrightarrow M$ proj. there exists $\gamma_C: C' \xrightarrow{\sim} C$ quasiisomorphism with C' cell. (dually for enough injectives & products $\Rightarrow$ localisation by co-cell).

Proof (colocalization): Construct

$$C'_{(n)} \xrightarrow{\gamma_{(n)}} C$$

epimorphism after applying homology

$H_i(\gamma_{(n)})$ is an epimorphism $\forall i$.

Choose $P_{(n)} \xrightarrow{\varphi_{(n)}} C'_{(n)}$ where $P_{(n)}$ is projective, 0 diff.

$H_i(\varphi_{(n)})$ is epimorphism to

$\operatorname{Ker} H_i(\gamma_{(n)})$

Set $C'_{(n+1)} = C\varphi_{(n)}$.

$C' = \operatorname{colim}(C_{(n)})$. $\square$

Example: Projective resolutions of an object of [an abl. category of] 1 are cell chain cxes.
(Injective resolutions are co-cell chain cxes.)

$$\cdots\cdots \to C_2 \to C_1 \to C_0 \to 0 \to 0 \sim \underline{C_i \text{ projective}}$$

is *always* a cell chain cx.

$$C_{(n)} := (C_n \to \cdots \to C_0)$$

$$P_{(n)} := C_{n+1}[n]$$

$$\varphi_{(n)} \left\{ \begin{array}{l} 0 \to C_{n+1} \to 0 \to 0 \cdots \qquad 0 \to 0 \\ \qquad \downarrow d \quad \downarrow \\ 0 \to C_n \to C_{n-1} \to C_{n-2} \qquad \to C_0 \to 0 \end{array} \right.$$

$$C\varphi_{(n)} \cong \quad 0 \to C_{n+1} \to C_n \to C_{n-1} \to C_{n-2} \qquad \to C_0 \to 0$$

A cell approximation of an object is achieved by a projective resolution (exists if there are enough projectives). (dually, for co-cell).

thought of as a chain cx. in degree 0.

Where do Tor and Ext come from?

There is a notion of a derived functor. Most generally, a functor $F : C \to D$ can be derived along any functor $\underbrace{G : C \to \bar{C}}_{\text{we are interested in } \Phi : C \to D_{\varepsilon} C}$.

derived category

a class of equivalences.

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "G"'] & D \\ \bar{C} \arrow[ur, dotted, "LF"'] \arrow[u, phantom, shift left=4ex, "\Uparrow"] & \end{tikzcd}$$

natural transformation
$LF \circ G \xrightarrow{\xi} F$

LF is called the left derived functor if it is universal with regard to this property.

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "G"'] & D \\ \bar{C} \arrow[ur, "LF"] \arrow[ur, bend right=60, "F'"'] \arrow[u, phantom, shift left=4ex, "\Uparrow"] & \end{tikzcd}$$

$F' \circ G \xrightarrow{\xi'} F$

$\exists!\ F' \xrightarrow[K]{} LF$

$\xi' = \xi \circ (K \circ G)$

(the right Kan extension)

Dually, RF:

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow \eta"] & D \\ \bar{C} \arrow[ur, dotted, "RF"'] & \end{tikzcd}$$

(co?) universal with this property.

<u>Theorem</u>: If C has co-localisation with respect to a class $\mathcal{B}$

$\gamma_X : X' \xrightarrow{\sim} X$

$X' \in \mathcal{B}$

Then the left derived functor of any functor $F : C \to D$ (with respect to $\Phi : C \to D_\xi C$) is naturally isomorphic to

$$LF(X) = F(X').$$

(dually for co-localisation). $\square$

<u>Example</u> : $\mathrm{Hom} : \mathcal{A}^{op} \times \mathcal{A} \longrightarrow \mathrm{Ab}$

functor of two variables

RHom with respect to the co-variable
or both variables

but in $\mathcal{A}^{op}$ it gets confusing

co-cell in $\mathcal{A}^{op}$ = cell in $\mathcal{A}$

<u>In particular</u>, for objects $M, N \in \mathrm{Obj}(\mathcal{A})$, one defines

$$\mathrm{Ext}^n_{\mathcal{A}}(M,N) := H^n(R\mathrm{Hom}(M,N))$$

Depending on whether $\mathcal{A}$ has enough projectives or injectives (or both), we can

① find a proj. resolution C of M

$$\mathrm{Ext}^n_{\mathcal{A}}(M,N) = H^n(\mathrm{Hom}_{\mathcal{A}}(C,N))$$

② find an injective resolution D of N

$$\mathrm{Ext}^n_{\mathcal{A}}(M,N) = H^n(\mathrm{Hom}_{\mathcal{A}}(M,D))$$

same answer!

HW ① Let $\Lambda[x] = \mathbb{C}\{1,x\}$, $x^2=0$. Consider the following

free module

chain cx of <u>free</u> $\Lambda[x]$-modules:

$$C: \cdots \to \Lambda[x] \xrightarrow{x} \Lambda[x] \xrightarrow{x} \Lambda[x] \xrightarrow{x} \cdots$$

$$1 \longmapsto x$$

$$x \longmapsto 0$$

Prove that C is not cell.