MATH 695 11/02/2022

R commutative ring (associative unital)

$$? \otimes_R ? : R\text{-Chain} \times R\text{-Chain} \to R\text{-Chain}$$

R ring

$$? \otimes_R ? : R\text{-Chain} \times R^{op}\text{-Chain} \to Ab$$

abelian groups

left R^{op}-modules = right R-modules

We can study $L\otimes_R$ (the left derived functor of $\otimes_R$)

(Theorem: It doesn't matter whether in one variable or both.)

In particular, if M, N are R-modules (R commutative)
or M left R-module N right R-module (R general)
Identify M, N with chain complexes in degree 0. Set

$$\mathrm{Tor}_i^R(M,N) = H_i\left(M \left(L\otimes_R\right) N\right).$$

R commutative ⇒ R-module
abelian group in general

left derived functor

The more immediate approach: For a projective resolution C of M,
projective res. D of N

$$\mathrm{Tor}_i^R(M,N) = H_i(C\otimes_R N) = H_i(M\otimes_R D) = H_i(C\otimes_R D).$$

$$\mathrm{Tor}_i^R(M,N) = \mathrm{Tor}_i^{R^{op}}(N,M)$$

For a projective resolution C of M resp. and injective resolution Q of N,

$$\mathrm{Ext}^i_R(M,N) = H^i(\mathrm{Hom}_R(C,N)) = H^i(\mathrm{Hom}_R(M,Q))$$
$$= H^i(\mathrm{Hom}_R(C,Q)).$$

<u>Note</u>: Some abelian categories have enough injectives but not enough projectives. Example: <u>Sheaves</u> ← study functions only partially defined (on open sets)

↑

ab. sheaf on space X

$\underbrace{\text{open sets of } X}_{\mathrm{Open}(X)}$, $\subseteq$ partially ordered

↓

category: $X \hookrightarrow Y$ when $X \subseteq Y$ ← only one morphism

Abelian Presheaf: A functor $\mathcal{F}: \mathrm{Open}(X)^{op} \longrightarrow \mathrm{Ab}$ ↙ $\mathcal{F}(U) \leftarrow$ an abelian group, elements are called <u>sections</u>.

$\mathcal{F}$ is a <u>sheaf</u> when it has the gluing property:

$$\underbrace{U = \bigcup U_i}_{\text{Open}}$$

$$\mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \underset{\text{restrict } j}{\overset{\text{restrict } i}{\rightrightarrows}} \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is an equaliser.

Abelian sheaves on X form an abelian category: $\mathrm{Ab}\text{-}\mathrm{Sh}(X)$.

(Forgetful functor: $\mathrm{Ab}\text{-}\mathrm{Sh}(X) \to \mathrm{Ab}\text{-}\mathrm{Presheaves}(X)$ has a left adjoint Sh called sheafification.)

Ker (or more generally, limits) in Ab-Sh(X) can be done section-wise

Coker (or more generally, colimits) done section-wise produce a pre-sheaf, sheafify

---

Ab-Sh(X) have enough injectives (but not enough projectives in general)

Sheaf cohomology: The constant sheaf $\underline{A} : \mathrm{Open}(X)^{op} \to \mathrm{Ab}$

A an abelian group

$U \longmapsto$ locally constant functions $U \to A$.

For an abelian sheaf $\mathcal{F}$ on X, one defines

$$H^i(X, \mathcal{F}) := \mathrm{Ext}^i_{\mathrm{Ab\text{-}Sh}(X)}(\underline{\mathbb{Z}}, \mathcal{F}).$$

e.g. CW-cxes

<u>Theorem</u>: If X is Hausdorff $\underbrace{\text{locally contractible}}$, then

$$H^i(X, \underline{A}) = \underbrace{H^i(X; A)}_{\text{singular cohomology}}$$

Class of examples: G be a discrete group.

$\mathbb{Z}[G] :=$ free ab. group on G, multiplication from G.

$\mathbb{Z} =$ trivial module $\quad \mathbb{Z}[G] \to \mathbb{Z}$

$\qquad\qquad\qquad\qquad\qquad\qquad g \mapsto 1$

For a $\mathbb{Z}[G]$-module M, one defines

$$H^i(G;M) = \mathrm{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z},M)$$

$$H_i(G;M) = \mathrm{Tor}_i^{\mathbb{Z}[G]}(\mathbb{Z},M).$$

When $G = \mathbb{Z}/k$, then a free ($\therefore$ projective) resolution of $\mathbb{Z}$ can be obtained as follows:

$$\cdots \xrightarrow{N} \mathbb{Z}[\mathbb{Z}/k] \xrightarrow{T} \mathbb{Z}[\mathbb{Z}/k] \xrightarrow{N} \mathbb{Z}[\mathbb{Z}/k] \xrightarrow{T} \mathbb{Z}[\mathbb{Z}/k] \qquad (*)$$

γ = generator of $\mathbb{Z}/k$. $\qquad T := 1-\gamma \in \mathbb{Z}[\mathbb{Z}/k]$

$N := 1+\gamma+\cdots+\gamma^{k-1}$

HW2: (a) Prove that $(*)$ is a (free) $\mathbb{Z}[\mathbb{Z}/k]$-resolution of $\mathbb{Z}$.

(b) Calculate $H^i(\mathbb{Z}/k;\mathbb{Z})$, $H_i(\mathbb{Z}/k;\mathbb{Z})$.