MATH 695 11/4/2022

<u>Products</u> in the cohomology of spaces come from the diagonal map

$$\Delta : X \longrightarrow X \times X$$
$$x \longmapsto (x,x)$$

co-commutative co-associative co-unital $\varepsilon : X \longrightarrow *$

On chain level, we have the Eilenberg-Zilber theorem

$$C(X\times Y)\xrightarrow[\psi]{\simeq} C(X)\otimes C(Y)$$

singular chains

unique up to natural chain homotopy

(Munkres)

(For cell chains, this is an canonical isomorphism). If X is a space,

$$C(X)\xrightarrow{C(\Delta)} C(X\times X)\xrightarrow[\simeq]{\psi} C(X)\otimes C(X).$$

One can dualise:

$$C^*(X;R)\otimes_R C^*(X;R)\longrightarrow C^*(X;R)$$

(R commutative ring).

$$C^*(X;R) \otimes_R C^*(X;R) \to C^*(X;R)$$

$$\cup : H^*(X;R) \otimes_R H^*(X;R) \to H^*(C^*(X;R) \otimes_R C^*(X;R)) \to H^*(X;R)$$

C, D chain complexes of R-modules: $H_* C \otimes_R H_* D \to H_*(C \otimes_R D)$

$[z] \otimes [t] \longmapsto [z \otimes t]$

because $dt = 0$

$\underbrace{[dz]}_{=0} \otimes [t] \longmapsto [dz \otimes t = \underbrace{d(z \otimes t)}_{=0}]$

$\cup$ on $H^*(X;R)$ makes $H^*(X;R)$ (for a commutative ring R) into a <u>graded-commutative</u> R-algebra

in modern writing, usually just denoted as $\cdot$

(A_n) is a $\mathbb{Z}$-graded ring

$$A_m \otimes A_n \to A_{m+n}$$

is graded-commutative if for $u \in A_m$, $v \in A_n$, we have

$$uv = (-1)^{mn} vu.$$

In algebraic topology, if you are trying to do commutative algebra on cohomology of spaces, we always have this sign to consider: super-commutative

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Not so easy to calculate $\cup$ directly, but it does give interesting rings:

Examples (to be justified later):

$$H^*(\mathbb{C}P^\infty;\mathbb{Z}) \cong \mathbb{Z}[b] \qquad \deg(b) = 2$$

generators

$$\begin{array}{ccccccccccc}
 & & & b^3 & & b^2 & & b & & \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & 0 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z} & \\
\cdots & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & \deg.
\end{array}$$

$$H^*(\mathbb{R}P^\infty;\mathbb{Z}) = \mathbb{Z}[b]/2b$$

$$\cdots \; 0 \;\; \mathbb{Z}/2 \;\; 0 \;\; \mathbb{Z}/2 \;\; 0 \;\; \mathbb{Z}/2 \;\; 0 \;\; \mathbb{Z}$$

We got the $\cup$ product from working on chain-level.
What about product in generalized cohomology? It cannot be there always: Even in ordinary cohomology, it is only there with coeffs. in a commutative ring

Fact: There is a product in K-theory. How do we get such information from "$\Omega^2 BU \times \mathbb{Z} \simeq BU \times \mathbb{Z}$".

$\boxed{\text{We need some analog of the "category of chain complexes" that would work for generalized (co)homology.}}$

The category of spectra (derived category unique, "point set" models vary)

? A more fundamental connection between homology and cohomology?

than, say, the universal coefficient theorem.

The concept of duality.

Alexander duality: The homology of a subspace of S^n is isomorphic to the cohomology of its complement (after handling base point properly).

nice

Precise statement: Let (S^n, X_+) be a based CW-pair.

n-sphere with a chosen base point.

Then

$$H_k(X;A) \cong H^{n-k}(S^n, S^n \setminus X; A).$$

Points to observe: 1. The homology of something = cohomology of something else.

Could we use this to pass between generalised homology and cohomology?

2. The proof has nothing to do with homology! It is purely space-level data $\rightsquigarrow$ duality on chain cxes.

trend on how to construct spectra.

HW 3 Prove that $\mathbb{C}P^2 \not\simeq S^4 \vee S^2$, even though the two spaces have the same homology. [Hint: You are allowed to use the stated result on the product in $H^*(\mathbb{C}P^\infty; \mathbb{Z})$. Deduce the product in $H^*\mathbb{C}P^2$, and prove that the product in $H^*(S^4 \vee S^2)$ is different.]