MATH 695 11/7/2022

Symmetric monoidal category $(\mathscr{C}, \otimes, \mathbb{1})$

commutative, associative, unital up to coherent natural $\cong$.

$\cong$ ~ equalities between words in different symbols in commutative monoids deduced from axioms

diagrams → do it two different ways.

(HW)(1): Using associativity, process

$((x \cdot y) \cdot z) \cdot t$

to

$x \cdot (y \cdot (z \cdot t))$

in two different ways. (e.g. you could get started by $((x \cdot y) \cdot z) \cdot t = (x \cdot (y \cdot z)) \cdot t$

or $((x \cdot y) \cdot z) \cdot t = (x \cdot y) \cdot (z \cdot t)$)

Write down the corresponding coherence diagram.

Strong duality We say that an object X of a symmetric monoidal category is strongly dual to an object Y if there are morphisms $\eta : 1 \to X \otimes Y$, $\varepsilon : Y \otimes X \to 1$ such that we have commutative diagrams

$$\begin{tikzcd} X \arrow[r, "\eta \otimes X"] \arrow[rr, bend right, "\mathrm{Id}_X"'] & X \otimes Y \otimes X \arrow[r, "X \otimes \varepsilon"] & X \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "Y \otimes \eta"] \arrow[rr, bend right, "\mathrm{Id}_Y"'] & Y \otimes X \otimes Y \arrow[r, "\varepsilon \otimes Y"] & Y \end{tikzcd}$$

We observe that this mimics the triangle identities of an adjunction, so it is equivalent to $? \otimes X$ being left (equivalently right) adjoint to $? \otimes Y$.

A symmetric monoidal category is called __closed__ if
$- \otimes X$ has a right adjoint $F(X, -)$

$$\mathrm{Mor}_{\mathcal{C}}(Z, F(X,Y)) \cong \mathrm{Mor}_{\mathcal{C}}(Z \otimes X, Y).$$

__Examples of closed symmetric monoidal categories:__

R-Mod (= R-modules for a commutative ring R)

R-Chain (even hR-Chain, DR-Chain)

__Note:__ hR-Chain, DR-Chain __do not__ have limits or colimits
(they do have products and coproducts)

Proposition: If $\mathcal{C}$ is a closed symmetric monoidal category and an object Y is strongly dual to an object X then

$$Y \cong F(X, 1)$$

(converse false)

Proof: $? \otimes X$ has right adjoint $F(X, ?)$ and also $? \otimes Y$.

Two right adjoints are isomorphic. In particular,

$$Y \cong 1 \otimes Y \cong F(X, 1). \qquad \square$$

Why is the converse false? If Y is strongly dual to X, then X is strongly dual to Y. $F(F(X,1),1) \cong X$.
(that's not always the case. Example: ∞-dimensional vector space.)

<u>Spanier-Whitehead duality</u>: (S^n, X_+) = CW-pair (some CW-structure on S^n)

$C(X)[n]$ is strongly dual to $C(S^n, S^n \setminus X)$

in h-Chain.

<u>Corollary</u>: $H_k(X) \cong H^{n-k}(S^n, S^n \setminus X)$.

$H^k(X) \cong H_{n-k}(S^n, S^n \setminus X)$.

Poincaré duality.

How do we prove Spanier-Whitehead duality? It has nothing to do with chain complexes (other than the Eilenberg-Zilber theorem). It is a statement about spaces. Could a finite CW-complex (= finitely many cells) have a strong dual in spaces? With respect to what operation? First idea: $\times$, but we also have pairs or based spaces, and the $\times$-product is not what we want there.

What we want is the $\wedge$-product ("smash-product" Frank Adams)

For based spaces X, Y, $X \wedge Y = X \times Y / (* \times Y) \cup (X \times *)$

$\boxed{\text{HW2} \quad \text{Prove that: } S^k \wedge S^l \cong S^{k+l}}$

In algebraic topology, the "wedge" is the 1-point union:

$X \vee Y = X \sqcup Y / *_X \sim *_Y$

X, Y based

$X_+ = X \sqcup \{*\}$

$(X_+) \wedge (Y_+) = (X \times Y)_+$

Compactly generated weakly Hausdorff spaces are a closed symmetric monoidal category with respect to $\wedge$, $F(X,Y) =$

$$= \{\text{continuous } \overset{\text{based}}{\text{maps}}\ X \to Y\}$$

(May: Concise course)

<u>The unit</u>: S^0.

This also shows that strong duality can virtually never occur in Based. If X connected based, $F(X, S^0) = *$.

the category of based spaces

We need to involve the higher spheres. "Make S^n into units".

We would work in $\mathrm{DBased}[\underbrace{? \wedge S^1}_{\text{based suspension } \Sigma}]^{-1}$

$$= \underbrace{\operatorname{colim}\left(\mathrm{DBased} \xrightarrow{\wedge S^1} \mathrm{DBased} \xrightarrow{\wedge S^1} \mathrm{DBased} \to \cdots\cdots \right)}_{\text{The Spanier-Whitehead category.}}$$

injective on objects

The Spanier-Whitehead category. ← does not have infinite product (or coproducts)

modern category of spectra has these

$$\Sigma = \wedge S^1 =: [1]$$

$$S[0] \vee S[-1] \vee S[-2] \vee \cdots$$