MATH 695 11/9/2022

(S^n, X_+) based CW-pair | Duality between $C(X)$

$C(S^n, S^n \setminus X)[-n]$

$\exists$ U open neighborhood of X in $S^n \setminus \{*\}$ $\quad S^n \setminus U \subset S^n \setminus X$ is a deformation retract

$X \subset U$ is a deformation retract

(Hatcher) $\quad (S^n, S^n \setminus U)$ CW-pair

$r: U \to X$

retraction

An even more fundamental setting:

$\tilde{C}(X)$, $\tilde{C}(S^n \setminus X)[1-n]$

Spanier function:

$$\mu: X \times (\mathbb{R}^n \setminus X) \to S^{n-1}$$

$$(x, y) \longmapsto \frac{x-y}{\|x-y\|}$$

Adams: Stable homotopy and generalized cohomology, Part III

$$\eta : S^n \longrightarrow X_+ \wedge (S^n / S^n \setminus U)$$

$$x \longmapsto \begin{cases} (\nu(x), x) & x \in U \\ * & x \in S^n \setminus U. \end{cases}$$

$$\varepsilon : (S^n / S^n \setminus U) \wedge X_+ \longrightarrow S^n = \mathbb{R}^n / \{ x \mid \|x\| \geq 1 \}$$

$$(y, x) \longmapsto \frac{1}{\lambda}(y - x)$$

λ-neighborhood of X is in U

<u>Theorem</u>:

$$\begin{tikzcd} S^n \wedge X_+ \arrow[r, "\eta \wedge Id_{X_+}"] \arrow[rr, bend right, "\simeq"', "\text{switch of coordinates}"] & X_+ \wedge (S^n / S^n \setminus U) \wedge X_+ \arrow[r, "Id_{X_+} \wedge \varepsilon"] & X_+ \wedge S^n \end{tikzcd}$$

$$\begin{tikzcd} (S^n / S^n \setminus U) \wedge S^n \arrow[r, "Id \wedge \eta"] \arrow[rr, bend right, "\simeq"', "\text{switch}"] & (S^n / S^n \setminus U) \wedge X_+ \wedge (S^n / S^n \setminus U) \arrow[r, "\varepsilon \wedge Id"] & S^n \wedge (S^n / S^n \setminus U) \end{tikzcd}$$

$\square$

695 notes from 2015 (lectures have titles by topic)

Immediate corollary - $C(X)$, $C(S^n \setminus X)[-n]$ are strong duals in h-chain.

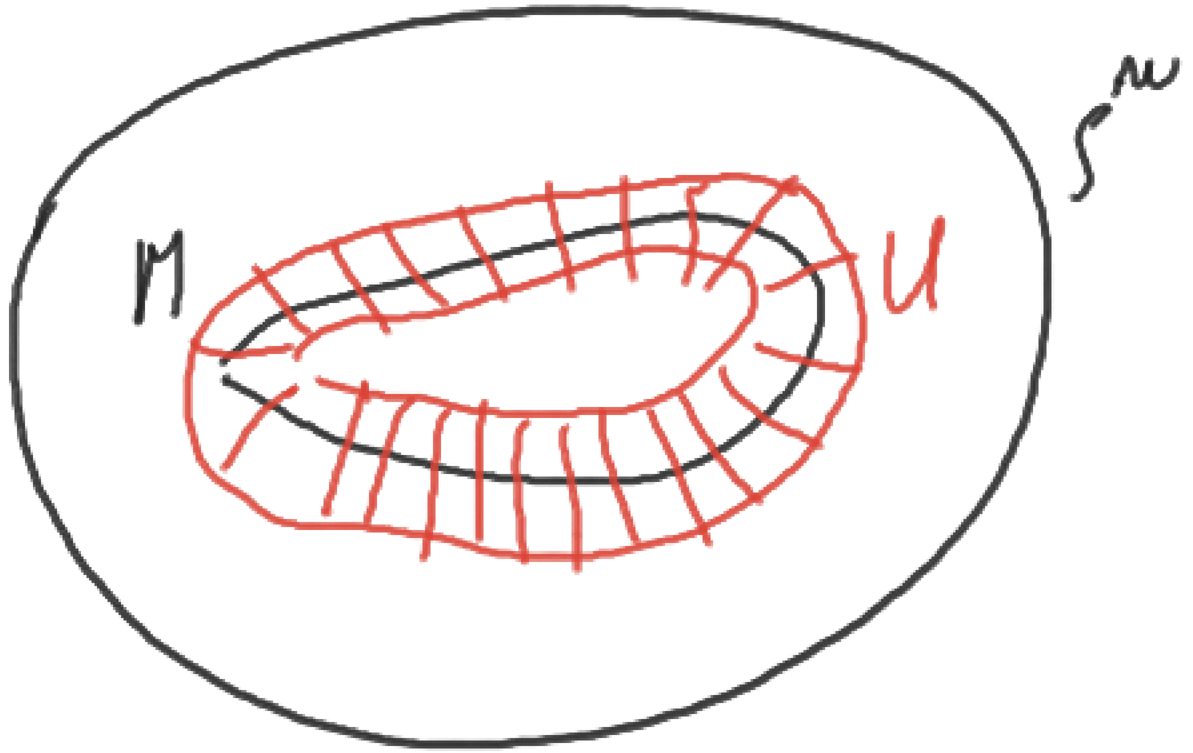
<u>Note</u>: $S^1 \overset{j}{\subset} S^3$ is a knot. $\pi_1(S^3 \setminus \mathrm{Im}\, j)$ almost determines the knot

For the purpose of Spanier - Whitehead duality, we do not distinguish between knots.

The most interesting case of duality: $X = M$ is a compact smooth manifold. (topological would do, even more general: Poincaré space)

Embed $M \subset S^n \setminus \{*\}$ as a (closed) smooth submanifold. $\exists$ tubular neighborhood U: $(U, M) \cong$ (total space of the normal bundle $\nu_M^{S^n}$, M)

What does this mean?



X be a topological space. A vector bundle is a finite-dimensional vector space $/X$ which is locally trivial.

over $\mathbb{R}$ (or $\mathbb{C}$)

$X \rightsquigarrow X_X$

all spaces homeo with a projection $\to X$

in my category

X is pullback over X

fiberwise product $/X$

vector spaces $/X$

stable under pullbacks.

locally trivial means $\forall x \in X$ $\exists U \ni x$ open pullback of my vector space $/X$ to U is $\cong U \times \mathbb{R}^n$ (resp. $U \times \mathbb{C}^n$)

(Milnor – Stasheff: Characteristic classes

Atiyah: K-theory)

HW 3 (The most basic problem in knot theory)

$$S^3 \cong (S^1 \times D^2) \cup (D^2 \times S^1)$$

Embed $S^1 \subset S^1 \times S^1 \overset{j}{\subset} S^3$

$z \mapsto (z^2, z^3)$.

(a) Compute $G = \pi_1(S^3 \setminus \mathrm{Im}\, j)$ Using Seifert-Van Kampen theorem.

(b) Construct an onto map $G \to \Sigma_3$ (symmetric group on 3 element) thus proving that G is not abelian and hence the knot is non-trivial.