MATH 695 11/11/2022

An n-dimensional real vector bundle over a space X is a $\mathbb{R}$-vector space in the category Top/X (fiberwise) which is locally constant

Objects: $Y \to X$

Morphisms:
$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \\ Y' \arrow[ur] & \end{tikzcd}$$

when pulled back to some open neighborhood U_x of every $x \in X$ becomes $\cong U_x \times \mathbb{R}^n \xrightarrow{\text{proj}} U_x$

<u>Examples</u>: For a smooth manifold M, the tangent bundle T_M.

over every point x, $(T_M)_x =$ tangent space to M @ x.

Different approaches to the definition — $M \subseteq \mathbb{R}^N$ smooth submanifold

$(T_M)_x$ tangent vectors to M at x

(Munkres – Analysis on manifolds)

$f: M \to N$ smooth: $Df: T_M \to T_N$

$U_x \subseteq M$ open $U_x \underset{\varphi_x}{\cong} \mathbb{R}^n$ ← coordinate system

Df identifies $T_{U_x} \cong f(U_x) \times \mathbb{R}^n$
$\downarrow$ proj
$f(U_x)$

$$\begin{tikzcd} \mathbb{R}^n & U_x \cap U_y \arrow[l, "\varphi_x"'] \arrow[r, "\varphi_y"] & \mathbb{R}^n \arrow[from=1-1, to=1-3, bend right, "\text{smooth}"'] \end{tikzcd}$$

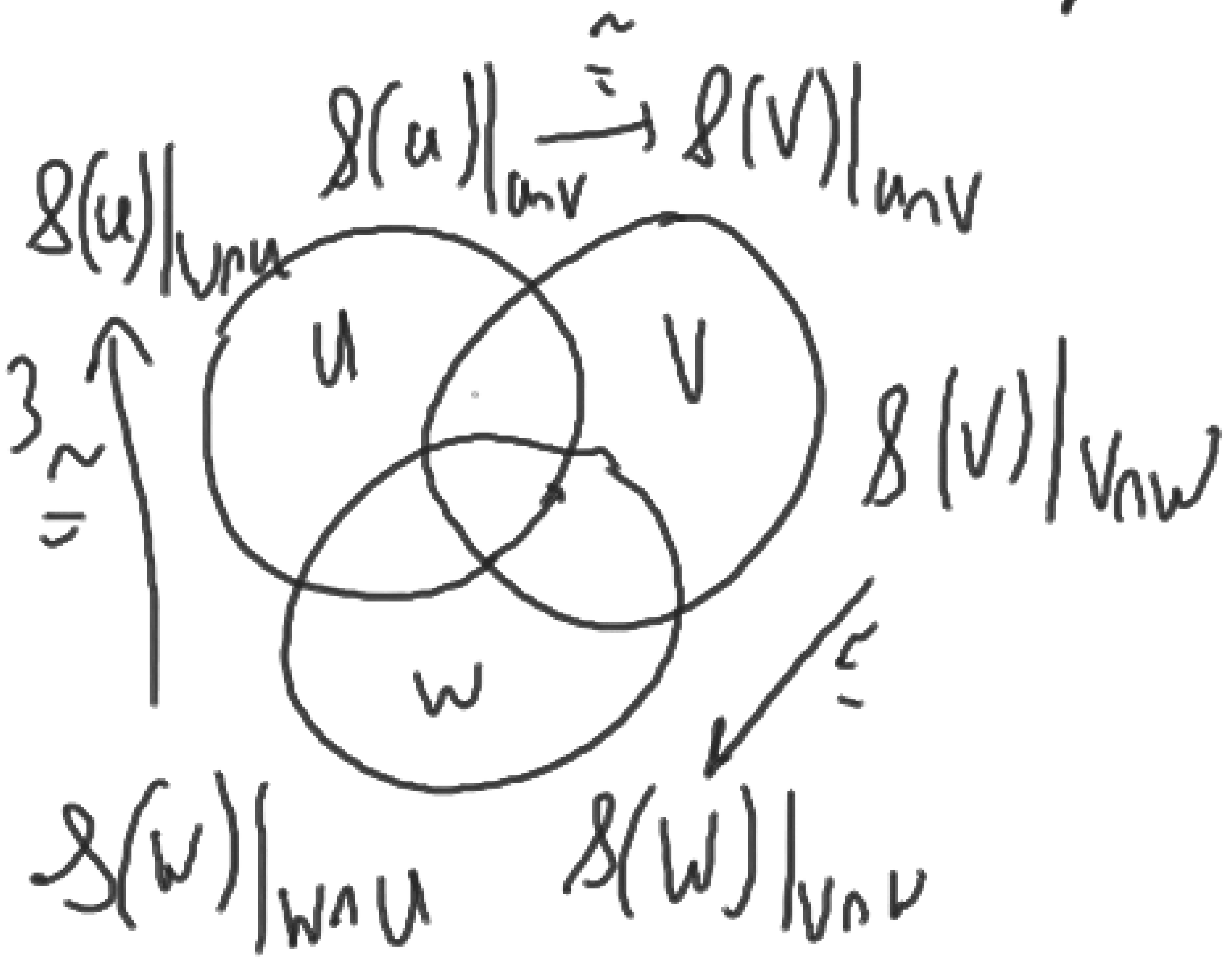
total diff. gives transformations between pullbacks of T_{U_x}, T_{U_y} to $U_x \cap U_y$ glue to get the tangent bundle

form a 2-category

Stack S similar to sheaf of sets on X $U \subseteq X \longmapsto S(U)$ ← a groupoid = category where every morphism is an isomorphism

- Is functoriality strict or up to natural $\cong$? (either way equivalent)

- gluing up to equivalence of categories
- on 3 natural $\cong$ of equivalences
- on 4 coherence diagram



This is why we can glue bundles from an open cover with appropriate gluing data – because they form a stack

For the theory of vector bundles: Atiyah: K-theory (Milnor-Stasheff: characteristic classes)

If X compact, ξ a vector bundle on X embeds into a trivial bundle $X \times \mathbb{R}^N$.

morphisms of bundles = morphisms of vector spaces / X (more than just $\cong$)

If X is paracompact, we can find a Euclidean metric on a bundle

continuously varying inner product (inner products from a convex set)

a bundle can also be thought of as the sheaf of its continuous sections (morphism of sheaf$)

injective on sheaf of sections

$(x,y) \mapsto (x, xy)$

$\mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{R}$

Over a compact space X, if ξ is an n-bundle $\exists$ m-bundle η such that $\xi \oplus \eta \cong X \times \mathbb{R}^N$ (trivial bundle)

$\times_X$ ("Whitney sum")

If M is a smooth manifold, $\exists$ η_M $\tau_M \oplus \eta_M \cong \underbrace{M \times \mathbb{R}^N}_{N \in \mathbb{N}}$

notation for trivial bundle

This is the (stable) normal bundle on M.

For $N >> 0$ η_M is unique up to $\cong$.

Spanier-Whitehead duality for manifolds

Let M be a compact smooth n-manifold (w/o boundary).
Then there exists an embedding

$$M \subset \mathbb{R}^N = S^N \setminus \{*\}$$

as a smooth submanifold.

For a bundle ξ over X, denote the total space
$E_\xi \to X$

$X \xrightarrow{\text{0-section}} E_\xi$

total space of the normal bundle of M

There exists a tubular neighbourhood U of M in S^N, such that $(U, M) = (E_{\nu_M}, M)$

Spanier - Whitehead duality tells us that

$$C(M) \text{ is dual to } \tilde{C}(S^N/S^N \setminus U)[-N]$$
$$\cong$$
$$C(S^N, S^N \setminus U)$$

$S^N \setminus U \subset S^N$ is a cofibration

in the category h-Chain.

Tubular neighborhood theorem $\Rightarrow$ $S^N/S^N \setminus U \cong$ 1-point compactification of E_{ν_M}

The Thom space M^{ν_M}.

if we had gone further, this would be true for generalised (co)homology

Poincaré duality!: Let $T_M \oplus \nu_M = N$ $(N >> 0)$ (M compact smooth manifold)

$$H_k(M;A) \cong \tilde{H}^{N-k}(M^{\nu_M};A)$$

(also can swap homology and cohomology)

HW 4 : On $\mathbb{RP}^n$, we have a $\underbrace{\text{1-dimensional real bundle}}_{\text{real line bundle}}$

γ which over every point $x \in \mathbb{RP}^n = S^n/x\sim -x$ is the line through x and the origin. Prove that the Thom space $(\mathbb{RP}^n)^\gamma$ is homeomorphic to $\mathbb{RP}^{n+1}$.