MATH 695 11/14/2022

If X is a compact space and ξ is an n-dim. real vector bundle on X, $p_\xi : E_\xi \to X$, the <u>Thom space</u> X^ξ is the 1-point

projection

total space

compactification of E_ξ. If X is not compact,

$$X^\xi := \underset{\substack{Z \subset X \\ \text{compact}}}{\mathrm{colim}}\ Z^\xi$$

we should really write $\xi|_Z$

(so we don't compactify X in the process)

HW ① Prove that if ξ is a trivial bundle ($E_\xi = X \times \mathbb{R}^n$) then $X^\xi \cong \Sigma^n X_+$.

So the idea is to think of the Thom space as a "twisted suspension" twisted by the bundle. If E is a generalised cohomology theory with a cup product (to be precisely defined soon, but we have an example, namely $H^*(?;R)$ where R is a commutative ring), we need to define a notion of E-orientable bundles, which would mean that E cannot tell the difference between ξ and the trivial bundle n.

(recall that the trivial bundle is denoted by the number)

How did we define the product on $H^*(?;R)$? Ingredient ① was the $\otimes$ of chain complexes (we are yet to generalise that to generalised cohomology). Ingredient ② was the *diagonal* $\Delta : X \to X \times X$.

the geometric input

For Thom spaces, we have the *Thom diagonal*. Let ξ be an n-bundle on X,

$X \amalg \{*\} =: X_+$

$$\theta : X^\xi \longrightarrow X^\xi \wedge X_+$$

$$z \in E_\xi$$

$$z \longmapsto (z, p_\xi(z)),$$

The point at ∞ is the base point

If E^* is a ring-valued generalised cohomology theory (Example: $H^*(?;R)$ with R a commutative ring) Then the Thom diagonal

$$\Theta : X^\xi \longrightarrow X^\xi \wedge X_+$$

induces a map

$$\Theta^* : \tilde{E}^*(X^\xi) \otimes E^*(X) \to \tilde{E}^*(X^\xi \wedge X_+) \xrightarrow{\tilde{E}^*\Theta} \tilde{E}^*(X^\xi).$$

In fact, this makes $\tilde{E}^*(X^\xi)$ into a module over the graded-commutative ring $E^*(X)$.

Thom's notion of orientability: A real vector n-bundle ξ on X called E-orientable (where E is a ring-valued generalized cohomology theory) if there exists a cohomology class

$$u \in \tilde{E}^n X^\xi$$

such that for every point $x \in X$, u restricts to a unit:

Explanation: Restrict ξ to $\{x\}$. $\{x\}^\xi \cong S^n$. not canonically

$$\{x\}^\xi \longrightarrow X^\xi$$

(functoriality)

$$\tilde{E}^n X^\xi \longrightarrow \tilde{E}^n \{x\}^\xi \cong \tilde{E}^n S^n \cong \tilde{E}^0 S^0 = \underbrace{E^0(*)}_{\text{commutative ring}}$$

$$u \longmapsto \text{unit} \in E^0(*)^\times$$

The assumption says that u restricts to an invertible element (= a unit) of $E^0(*)$.

<u>Theorem</u> (Thom isomorphism theorem): If ξ is an E-orientable real vector n-bundle then

$$\theta^*: \tilde{E}^n X^\xi \otimes E^k X \longrightarrow \tilde{E}^{k+n} X^\xi$$

restricted to $u \otimes ?$ defines an isomorphism

$$\theta_u^*: E^k X \to \tilde{E}^{k+n} X^\xi.$$

(The Thom isomorphism.)

in place of X

Proof: First true for coordinate patches U_i of ξ (i.e. where the bundle is trivial), as well as for all open subsets of U_i. (After all, it is true for trivial bundles) Then, by induction, it is true for open subset of $X_{i_1} \cup \dots \cup X_{i_k}$ (using the Mayer–Vietoris sequence)

U, V open. $\cdots \to E^k(U \cup V) \to E^k(U) \oplus E^k(V) \to E^k(U \cap V)$

$$\begin{array}{c} E^k(U \cap V) \\ \downarrow d \\ E^{k+1}(U \cap V) \\ \vdots \end{array}$$

Then it is true for X by the limit axiom in cohomology. $\square$

Back to manifolds: let M be a compact smooth n-manifold.
Let E be a ring-valued generalised cohomology (example: $H^*(?;R)$, R comm. ring).

We say that M is E-orientable if T_M is E-orientable (equivalently, ν_M is E-orientable).

Suppose M (as above) is E-orientable. $M \subset \mathbb{R}^N$

normal bundle in $\mathbb{R}^N$, using tubular neighborhood theorem

$$E_k(M) \cong \tilde{E}^{N-k}(M^{\nu_M^{\mathbb{R}^N}}) \cong \tilde{E}^{n-k}(M)$$

dimension: $N-n$

This is known as Poincaré duality (for generalised cohomology).

For now, all this is rigorous for $E = H^\vee(?;R)$ R commutative ring. What does orientability with respect to these theories mean?

$$R = \mathbb{Z}, \mathbb{Q}, \mathbb{R} \leftarrow \quad H(?;R)\text{-orientability} = \text{classical orientability}$$

(∃ nowhere vanishing de Rham n-forms)

$$R = \mathbb{Z}/2 \quad \leftarrow \quad H(?;\mathbb{Z}/2)\text{-orientability is always true.}$$

$$(\mathbb{Z}/2^\times = \{1\})$$